PREFORM AND DIE DESIGN PROBLEMS IN METAL FORMING

A Dissertation

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by

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A methodology is presented for designing the die and preform for metal forming operations, by posing them as optimization problems for minimizing certain error norms, where the error norm is a measure of the deviation of the actual state from the desired state.

For solving the large deformation metal forming problem, the material is modeled as a hyperelastic, viscoplastic solid. A constitutive model with a single scalar variable representing the isotropic resistance to the plastic flow is employed. The contact between the die and the workpiece is described in the continuum setup. A principle of virtual work equation is developed for the motion of the body including the contact and the friction using an augmented Lagrangian formulation. The principle of virtual work equation is solved by using a finite element method.

Sensitivity fields are defined rigorously for a variation in the shape of the workpiece as well as the process parameters such as the die shape. Then, the governing equations for the sensitivity fields are developed in a continuum form. The sensitivity field equations are solved by using the same spatial and temporal discretization
as the large deformation mechanics problem (“direct problem”). The design space for performing the optimization for die and preform design is chosen as finite dimensional. A modified BFGS method is used to perform the optimization, for which the gradient fields are obtained from the sensitivity fields.

Several example problems are worked out for demonstrating the direct simulation of the metal forming problems. An example problem each is worked out for demonstrating the die design and the preform design methodology.
BIOGRAPHICAL SKETCH

The author was brought up in a little town in the Southern Indian state of Tamil Nadu called Nagapattinam on 30th June, 1970. After completing his Higher Secondary Education from National High School, Nagapattinam, the author received his undergraduate degree in Mechanical Engineering from the Indian Institute of Technology, Madras in August 1991. He joined the Sibley School of Mechanical and Aerospace Engineering, Cornell University for his graduate studies in the Fall of 1991.

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Chapter 1

Introduction

Metal forming processes account for a significant portion of the industrial production of metal components. The primary objective of a metal forming operation is to induce a desired change of shape on a piece of metal. In a given forming process, the material state and geometry of the final product depend on several process parameters such as loading conditions, die surfaces, die lubrication conditions, geometry of the initial workpiece, material properties of the initial workpiece and the evolution of those material properties.

A very important concern in any forming process is the design of the process in such a way that a final product with a desired material state and geometry is achieved. The design of forming processes is considered as the design of the initial workpiece and of the subsequent shapes at each of the forming stages known as preforms as well as the design of the dies at each stage that will deform the workpiece at each stage to the one for the next stage. A overall process design
involves the entire design of the sequence of operations, the die shapes for each stage, and the initial preform shape.

This work focusses on defining the die design and the preform design problems in a rigorous manner and providing a methodology to solve them.

Die and preform designs in the metal forming industry usually followed some empirical guidelines found in design handbooks [1], [2]. These methods required building a prototype die and preform and testing them for the performance, even for the known shapes. These methods were severely restricted as they could not be used for new materials with different properties, and newer product shapes.

With the development of the computer simulations of the metal forming processes using methods such as the finite element method, empirical design procedures using computer simulations were being developed. A systematic study of some of the die and preform design problems was done by Kobayashi and colleagues ([3] - [7]). They introduced the so called ‘backward tracing technique’ and “traced backward the loading path in the actual forming process from a given final configuration.” However, plastic deformation being an irreversible process, no natural path really exists from the final workpiece shape to the initial workpiece. The backward tracing approach by Kobayashi and colleagues depended on the prior knowledge of the forming processes and approximate die shapes and preform shapes for the process under design. Han et al. [8] improved on this backward tracing method by introducing a backward deformation optimization method. In this method, some constraints are placed on the “reverse deformation” of the material such as minimizing the standard deviation of the effective strain rate distribution to result in
an optimum reverse path.

The preform design and the die design problems can be formulated under a rigorous mathematical basis by posing them as optimization problems [9]. The objective function for these optimization problems can be defined as an appropriate measure of the error between the desired final state and the numerically calculated state for a given set of process variables. Such a rigorous approach does not make any assumptions about the “reverse paths.” In the path dependent elasto-plastic problems, the final destination is achieved by several initial workpieces and the corresponding process conditions. Thus, we are only interested in picking the optimal preform and die shapes that will achieve a certain specific objectives we have in mind.

In order to solve such optimization problems, one usually employs a sequential search method starting from a reference solution. Such sequential search procedures require the calculation of the gradients of the objective function with respect to the process parameters. The evaluation of the gradient of the objective function with respect to the process parameters can be extremely complicated and computationally intensive. If the decision variables are infinite dimensional functions, as for example the shape of the die in a die design problem, evaluating the gradient can be a difficult task. The practice in the literature seems to be that of approximating the process parameters with an appropriate superposition of selected basis functions and then evaluating the gradient in the finite dimensional subspace.

Sensitivity analysis is the method widely used for calculating the gradient of the objective function and for studying the effects of process parameters in the state
of the final product. In the literature, the sensitivities are usually being calculated using (a) finite differences approximations, (b) the adjoint variable method or (c) the direct differentiation method ([20], [19]). An extensive review of various sensitivity methods employed in the literature is presented in Tortorelli and Michaleris [10]. In the finite differences method, the direct problem is solved for two different (but nearby) set of process conditions and the difference of the solutions obtained is used to calculate the sensitivities. Employing finite differences to evaluate the gradients can be a waste of computer resources and time as one has to solve the direct nonlinear problem many number of times. Furthermore, the resulting solution could be quite inaccurate if the actual sensitivities are of the order of the numerical errors. Tortorelli et al. ([11], [12]) used an adjoint variable method for obtaining the design sensitivities in heat transfer problems. However, there are significant difficulties in employing this method for calculating the sensitivities in history dependent problems [13].

In the direct differentiation method, a set of field equations are developed by considering the variation of the field equations of the direct problem with respect to small changes in the process parameters. The sensitivity field equations are linear and hence it takes much less time to numerically solve them in comparison to the finite differences approach. Furthermore, a high degree of accuracy can be expected.

A fair amount of work has been done in the sensitivity analysis of plasticity problems ([13] - [18]). Using the direct differentiation method, Lee et al. [19] developed the sensitivity equations and employed explicit integration schemes to integrate the constitutive equations. Vidal and Haber [20] developed the sensitivity
field equations for hypoelastic rate-independent plasticity for small deformations and used an implicit integration scheme.

There are many advantages in developing an infinite dimensional scheme for solving design forming problems, the obvious one being that of achieving an optimal solution that could otherwise be missed out while using a particular finite dimensional representation of the unknown process parameters. The numerical solution of these infinite dimensional optimization problems requires that eventually a finite dimensional discretization of the working design space is performed. However, one still achieves better results when compared to the numerical results of finite dimensional optimization schemes ([22], [23] and [24]). For developing an infinite dimensional optimization scheme, one needs to develop a sensitivity analysis where the design variables belong to an infinite dimensional space. In traditional literature on sensitivity analysis, the sensitivity of any quantity is simply defined as the partial derivative of that quantity with respect to the design variable. However, while dealing with functionals of tensor fields (e.g. functionals involving the material state), one must be very precise and careful in the definition of sensitivities.

In the subsequent chapters, we present the model and solution scheme used for the direct metal forming problems. Following that, we present the definitions of parameter sensitivities and shape sensitivities. Then, we develop the parameter sensitivity and shape sensitivity equations for the mechanics problem, and provide a methodology for solving the die design problems and preform design problems. We provide a simple example for each of the design problems to demonstrate these methods. We conclude with suggested developments on the current work.
Chapter 2

Large Deformations of

Hyperelastic Viscoelastic Solids

2.1 Introduction

Metal forming processes account for a significant portion of the industrial production of metal components. The primary objective of a metal forming operation is to induce a desired change of shape on a piece of metal. The major concerns in any metal forming process are the estimation of the forces required to form the product, and the material property of the final product. Accurate predictions of the forming forces required and the changes in the material properties during a process are possible by mathematically modeling the material and the process, and numerically solving the mechanics problem.

There are several complex, mathematical issues involved in modeling a metal
forming process. A substantial body of work has appeared in the literature dealing with the various aspects of these complex issues. The main purpose of this particular work is to propose a scheme for designing a metal forming process more efficiently. Since a systematic study of such design problems is being attempted for the first time, we have made several simplifying assumptions about the material model and the deformation process.

In this chapter, we present a brief outline of the modeling of the elasto plastic deformation of the material and the numerical solution scheme employed to solve these metal forming problems.

Metal forming processes involve large elasto-plastic deformation of the material, typically under the action of a die. Usually, the elastic deformations are extremely small and the plastic deformation, very large. The deformations are usually anisotropic. The process may occur at an elevated temperature or at the room temperature. During the process, there is always an increase in the temperature of the workpiece due to the dissipation of the plastic work. In this work, however, we model the process as isothermal and the material deformation as isotropic.

The plastic flow due to the motion of the dislocation is inherently rate-dependent [28]. Therefore, following Anand [25], we model the plastic deformation as rate-dependent viscoplastic, and the elastic deformation as hyperelastic. We assume that the instantaneous response of the material is determined by the current values of the Cauchy Stress and a scalar state variable representing the isotropic resistance to plastic flow. An extensive review of the analysis of large deformation mechanics problems can be found in [28] - [31].
One of the major difficulties in the numerical simulation of the metal forming processes is the modeling of the contact and the friction at the die-workpiece interface. Based on the works of Simo and Laursen [33] - [36], we model the die-workpiece interface to obey Coulomb friction law with a constant coefficient of friction. In this work, we develop and solve the principle of virtual work (PVW) form of the equilibrium equations involving contact and friction by employing an augmented Lagrangian method.

The large deformation problem is solved using an updated Lagrangian reference frame setup. An implicit time integration scheme is employed to integrate the evolution equations. In the following sections, a framework is presented for the kinematic and the constitutive modeling. Then, the principle of virtual work equations are developed, and a complete linearization of the PVW equations is presented. A finite element program has been developed to solve the large deformation metal forming problems. A brief outline of the developed code is presented here and a few example problems in axisymmetric and plane strain cases are worked out to show the effectiveness of this method in solving large deformation, frictional contact problems.

2.2 Kinematic and constitutive equations

2.2.1 Kinematics

Let us consider the motion of a body occupying the configuration $B_0$ at time $t = 0$ under the action of external forces. The motion of the body is represented by a
smooth mapping $\hat{x}(\cdot, \cdot)$ such that, at any given time $t$, the location $x$ of a material point $p$, which at time $t = 0$ occupied the position $X$, is given by

$$x = \hat{x}(X, t)$$ (2.1)

The total deformation gradient $F$ is the derivative of the current position with respect to the position field in the reference configuration $B_0$, and is given by,

$$F = \nabla \hat{x}(X, t) = \frac{\partial \hat{x}(X, t)}{\partial X}$$ (2.2)

We require that $\det F > 0$ for the motion of the body to be meaningful. The total deformation gradients at times $t_n$ and $t_{n+1}$ are $F_n$ and $F_{n+1}$ respectively, as shown in Figure (2.1). In the Lagrangian analysis, the motion of each particle is followed as the particle occupies different configurations in the space at different times, and
the deformation gradients are computed with respect to a reference configuration \( B_0 \). In the case of the updated Lagrangian analysis, the reference configuration is updated continuously. Thus, in an analysis involving a sequence of time indexes \((0, t_1, t_2, \cdots, t_n, t_{n+1}, \cdots)\), the reference configuration will be \( B_n \) when the current configuration becomes \( B_{n+1} \). Thus, we define the relative deformation gradient \( F_r \) as,

\[
F_r = F_{n+1}(F_n)^{-1}
\]  

(2.3)

### 2.2.2 Equilibrium equations

In this work, we assume that the processes are quasi-static and the inertial effects can be ignored entirely. Thus, the body is under equilibrium all the time. Let \( T \) be the Cauchy stress and \( b \), the body force field defined on the current configuration. Then, at a time \( t_{n+1} \), the equilibrium equations can be written as

\[
\nabla_{n+1} \cdot T + b = 0 \quad \forall \, x \in B_{n+1}
\]  

(2.4)

The subscript \((n+1)\) indicates that the divergence of \( T \) is described in the current configuration \( B_{n+1} \). The equilibrium equations can also be expressed in the reference configuration \( B_n \) as,

\[
\nabla_n \cdot P_r + f_r = 0 \quad \forall \, x \in B_n
\]  

(2.5)

where \( P_r \) and \( f_r \) are obtained as,

\[
P_r = \det F_r \, T \, F_r^{-T}
\]

\[
f_r = \det F_r \, b
\]  

(2.6)
In order to solve the equilibrium equations, certain boundary conditions have to be specified at all times on the boundaries of $B$. In addition, the constitutive relationship connecting the Cauchy Stress $T$ and the deformation gradient $F$ should be specified. In subsection 2.2.3, the constitutive equations for the material under consideration will be described. Section 2.3 will detail the numerical procedure for solving the equilibrium equations with appropriate boundary conditions.

### 2.2.3 Constitutive equations

The most commonly used phenomenological model for infinitesimal elastic-plastic deformation is the classical $J_2$ flow theory with isotropic hardening. The set of constitutive equations considered here are essentially a generalization of this theory to model large, finite elasto-plastic deformations and rate dependent plastic flow.

The total deformation gradient $F$ is assumed to be decomposed as,

$$F = F^e \tilde{F}^p$$  \hspace{1cm} (2.7)

where, $F^e$, the elastic deformation gradient and $\tilde{F}^p$, the plastic deformation gradient (Figure (2.2)). We further assume that the plastic deformation is isochoric, and hence,

$$\det \tilde{F}^p = 1$$

$$\det F^e > 0$$  \hspace{1cm} (2.8)

Here, the intermediate destressed configuration is not necessarily a physically achievable one. Rather, it is an imaginary configuration that aids in visualising
the individual mechanisms that manifest in plastic and elastic deformations. From the undeformed state, we assume that the intermediate configuration is achieved by means of slip and glide of the dislocation planes. From the intermediate configuration, the actual configuration is achieved by means of elastic rotations and stretches.

Following Anand [25], the constitutive equations for the stresses are written as,

$$\tilde{T} = L^{\varepsilon} \left[ \tilde{E}^{\varepsilon} \right]$$

(2.9)

where $\tilde{E}^{\varepsilon}$ the strain measure, is defined with respect to the intermediate (unstressed) configuration as

$$\tilde{E}^{\varepsilon} = \ln \tilde{U}^{\varepsilon}$$

(2.10)

The corresponding conjugate stress measure $\tilde{T}$ is the pullback of the Kirchhoff stress
with respect to \( R^e \),

\[
\tilde{T} = R^{eT} \det(U^e) \ T \ R^e
\]  

(2.11)

Here \( U^e \) and \( R^e \) are calculated from the polar decomposition of \( F^e \),

\[
F^e = R^e U^e
\]  

(2.12)

For an isotropic material, the elastic moduli \( \mathcal{L}^e \) are given by

\[
\mathcal{L}^e = 2\mu \ I + \left( \kappa - \frac{2}{3} \mu \right) I \otimes I
\]  

(2.13)

where \( \mu \) is the shear modulus and \( \kappa \) is the bulk modulus. A flow rule is given for the evolution of \( \dot{F}^p \) as follows:

\[
\ddot{L}^p = \dot{F}^p \left( \dot{F}^p \right)^{-1} = \sqrt{\frac{3}{2}} \ \dot{\varepsilon}^p \ \tilde{N}^p(\tilde{T}', \tilde{\sigma})
\]  

(2.14)

where

\[
\tilde{N}^p(\tilde{T}', \tilde{\sigma}) = \sqrt{\frac{3}{2}} \ \tilde{T}'
\]  

(2.15)

\[
\tilde{T}' = \tilde{T} - \frac{\text{tr} \ \tilde{T}}{3} \ I
\]  

(2.16)

and

\[
\tilde{\sigma} = \sqrt{\frac{3}{2}} \ \tilde{T}' \cdot \tilde{T}'
\]  

(2.17)

We also make the assumption that an intermediate configuration can be so chosen in case of isotropic materials such that the skew symmetric part of the velocity gradient (spin) vanishes identically. This results in,

\[
D^p = L^p
\]

\[
\tilde{W}^p = 0
\]  

(2.18)
The evolution of the equivalent plastic strain \( \dot{\varepsilon}^p \) is specified via uniaxial experiments as:

\[
\dot{\varepsilon}^p = f(\bar{\sigma}, s)
\]  

(2.19)

while the evolution of the isotropic isothermal scalar resistance \( s \) is also obtained from experiments in the form,

\[
\dot{s} = g(\bar{\sigma}, s)
\]  

(2.20)

### 2.3 The initial and the boundary value problem

The kinematic and the constitutive equations, together with the equilibrium equations and the initial and the boundary conditions, completely describe the motion of the body. In this section we will develop a numerical solution procedure to solve this mechanics problem. We divide the problem into two: (1) a constitutive problem and (2) a kinematic problem. In the former, we try to determine the stresses and the state variables, if the entire deformation history is known. In the latter, we try to determine the deformation history itself, if the current stresses and the state variables of the body, and the external forces acting on the body are known.

#### 2.3.1 Implicit time integration algorithm

Let us suppose that the history of the deformation is completely known and that we are interested in evaluating the history of the stresses and the state variables. To evaluate the stresses and the state variables, the evolution equations must be integrated over the time domain. These evolution equations have been known to be
numerically very stiff. In case of large deformation problems, using Euler forward time-integration procedure will lead to excessive number of solution increments [28]. Therefore, we implement an implicit time integration scheme based on [29].

In the following, we provide a method to calculate the stresses and the state variables at time $t_{n+1}$, given the configurations at time $t_n$ and $t_{n+1}$, and knowing the stresses and the state variables at time $t_n$.

Employing Euler backward integration to solve the differential equations (2.14) and (2.20), we obtain:

\[
F_p^{n+1} = \exp \left( \frac{\Delta t}{2} \tilde{D}_p^{n+1} \left( \tilde{T}_{n+1}' \right) \right) F_p^n \tag{2.21}
\]

and

\[
s_{n+1} - s_n - \Delta t \ g(\tilde{\sigma}_{n+1}, s_{n+1}) = 0 \tag{2.22}
\]

To solve these algebraic equations, we introduce a trial elastic deformation gradient $F_e^a$ defined as,

\[
F_e^a = F \ F_e^n \tag{2.23}
\]

where $F$ is defined in the equation (2.3). Now, we have,

\[
\tilde{T}_a = \mathcal{C} \left[ \tilde{E}_e^a \right] \tag{2.24}
\]

$\tilde{\sigma}_a$ is defined from the deviatoric part of the trial stress as,

\[
\tilde{\sigma}_a = \sqrt{\frac{3}{2} \tilde{T}_a' \cdot \tilde{T}_a'} \tag{2.25}
\]

and $\bar{p}_a$ as,

\[
\bar{p}_a = -\frac{1}{3} \text{tr} \left( \tilde{T}_a \right) \tag{2.26}
\]
It can easily be shown (Appendix A) that the tensors $\tilde{T}_n$ and $\tilde{T}_{n+1}$ are in the same direction. So, we only need to evaluate the scalar quantities $\tilde{\sigma}_{n+1}$, $\tilde{p}_{n+1}$ and $s_{n+1}$.

Based on Appendix A, one can also show that,

$$\tilde{p}_{n+1} = \tilde{p}_s$$

and

$$\tilde{\sigma}_{n+1} - \sigma_s + 3\mu \Delta t f(\tilde{\sigma}_{n+1}, s_{n+1}) = 0$$

Solving the nonlinear, scalar, algebraic equations (2.28) and (2.22) together, one can obtain the values of $\tilde{\sigma}_{n+1}$ and $s_{n+1}$. From the values of $\tilde{\sigma}_{n+1}$, $\tilde{p}_{n+1}$ and $s_{n+1}$, one can calculate $T_{n+1}$ and $F^e_{n+1}$. A summary of the time-integration constitutive algorithm is presented below:

1. Calculate the trial elastic deformation gradient

$$F^e_* = F_r F^e_n$$

2. Perform the polar decomposition

$$F^e_* = R^e_* U^e_*$$

3. Compute the trial elastic strain $\tilde{E}^e_*$. To do this, first compute the spectral decomposition of $U^e_*$

$$U^e_* = \sum \lambda^i_* e^i_* \otimes e^i_*$$

where $\lambda^i_*$ are the eigen values and $e^i_*$ the eigen vectors of $U^e_*$.

$$\tilde{E}^e_* = \sum \left( \ln \lambda^i_* \right) e^i_* \otimes e^i_*$$
4. Calculate the trial stress
\[ \bar{T}_s = \mathcal{L}^e \left[ E_s^e \right] \]

5. Calculate the mean normal pressure and the deviatoric part of \( \bar{T}_s \)
\[ \bar{\rho}_s = -\frac{1}{3} \text{tr} \left( \bar{T}_s \right) \]
\[ \bar{T}_s' = \bar{T}_s + \bar{\rho}_s \mathbf{I} \]

6. Calculate the trial equivalent tensile stress \( \bar{\sigma}_s \)
\[ \bar{\sigma}_s = \sqrt{\frac{3}{2} \bar{T}_s' \cdot \bar{T}_s'} \]

7. Calculate \( \bar{\sigma}_{n+1} \) and \( s_{n+1} \) by solving the following equations (Appendix B).
\[ \bar{\sigma}_{n+1} - \bar{\sigma}_s + 3\mu \Delta t f \left( \bar{\sigma}_{n+1}, s_{n+1} \right) = 0 \]
\[ s_{n+1} - s_n - \Delta t g \left( \bar{\sigma}_{n+1}, s_{n+1} \right) = 0 \]

8. Obtain the radial return factor \( \eta_{n+1} \)
\[ \eta_{n+1} = \frac{\bar{\sigma}_{n+1}}{\bar{\sigma}_s} \]

9. Update the stress \( \bar{T}_{n+1} \)
\[ \bar{T}_{n+1} = \eta_{n+1} \bar{T}_s' - \bar{\rho}_s \mathbf{I} \]

10. Transform \( \bar{T}_{n+1} \) to \( T_{n+1} \)
\[ T_{n+1} = \exp \left( \frac{\bar{\rho}_s}{\kappa} \right) R_s^e \bar{T}_{n+1} R_s^T \]

11. Update \( F_{n+1}^e \)
\[ F_{n+1}^e = R_s^e \left( \sum \frac{\left( \lambda_i^j \right)^{\eta_{n+1}}}{\left( \lambda_1 \lambda_2 \lambda_3 \right)^{\left( \eta_{n+1} - 1 \right) / 3}} e_i^j \otimes e_i^j \right) \]
2.3.2 Boundary conditions

In order to completely solve the deformation problem described in the previous section, the boundary conditions have to be prescribed at each time instant. The boundary conditions on regions of the boundary could be of one of the following type:

1. Normal and tangential displacement field is known.

2. Normal and tangential traction field is known.

3. Normal displacement and tangential tractions or vice-versa are known.

4. Implicit contact conditions are specified.

The first three types of boundary conditions are explicit in nature and can be applied directly in solving the principle of virtual work equation. In the following section, we will consider the contact condition between the die and workpiece.

2.3.3 Contact and friction in a die-workpiece interface

We model the contact and the friction following the scheme developed by Simo and Laursen ([33] – [36]). We consider dies to be rigid. We only consider problems in plane-strain and axi-symmetric cases. The die \( D \) is parametrized in two dimensions using a parameter \( \xi \) and the functions \( y(\xi) = (y_1(\xi), y_2(\xi)), 0 \leq \xi \leq 1 \). We define a fixed right-handed basis \((e_1, e_2, e_3)\), with \( e_3 \) coming out of the plane of the paper (as shown in Fig. (2.3)) and a convected basis \((r, n, e_3)\) at each point defined by a
Figure 2.3: Representation of the die surface and definition of the gap function. 

![Diagram showing the die surface and gap function representation.](image)

A particular value of $\xi$ using the parametric representation of the die. The tangent vector $\tau$ is given by

$$\tau_1 = y \xi = \frac{\partial y_1}{\partial \xi} e_1 + \frac{\partial y_2}{\partial \xi} e_2$$  \hspace{1cm} (2.29)

The unit tangent vector $r$ is given by

$$r = \frac{\tau_1}{\| \tau_1 \|}$$  \hspace{1cm} (2.30)

The unit normal vector is obtained as

$$n = e_3 \times r$$  \hspace{1cm} (2.31)

where $\times$ denotes the vector cross product. Any Vector $a$ in the direction of $\tau_1$ is represented as,

$$a = a^\dagger \tau_1 = a r$$  \hspace{1cm} (2.32)

where, $a^\dagger \| \tau_1 \| = a$.

The die separates the space into an “admissible region” and an “inadmissible region.” The parametric form of the die is written in such a way that the normal
vector $\mathbf{n}$ is pointing into the inadmissible region. Then, we can define the gap function $g$ of any point $\mathbf{x}$ in space as the shortest distance of that point from the die. The sign of the function $g$ is positive if $\mathbf{x}$ belongs to the inadmissible region and negative otherwise. Thus, we write:

$$\mathbf{x} - \mathbf{y} = g(\mathbf{x}) \mathbf{n}(\mathbf{y})$$

(2.33)

where $\mathbf{y} \in \mathcal{D}$ is the value of $\mathbf{y}$ that minimises the norm, $|\mathbf{x} - \mathbf{y}|$. A unique value of the parameter $\xi$ is associated to each $\mathbf{y}$.

With this definition of the gap function, we can write down the impenetrability constraints as below:

For all $\mathbf{x}_n \in B_n$, with $\mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{u}(\mathbf{x}_n)$,

$$g(\mathbf{x}_{n+1}) \leq 0$$

$$t_N = -\mathbf{n} \cdot (\mathbf{P}, \mathbf{m})$$

$$t_N g(\mathbf{x}_{n+1}) = 0$$

(2.34)

where $t_N$ is the normal contact pressure. The negative sign in the definition of $t_N$ is chosen so that the number $t_N$ remains positive. The vector $\mathbf{P}, \mathbf{m}$ is the current traction, defined in the domain $B_{n+1}$.

We assume that Coulomb friction exists in the contact region with a coefficient of friction $\mu$. The tangential velocity of the workpiece relative to the rigid die is written (using equation (2.29)) as

$$\mathbf{v}_T = \frac{d}{dt}[\dot{\mathbf{y}}] = \mathbf{y}_{\xi} (\dot{\xi}) \hat{\xi} = \hat{\xi} \tau_1$$

(2.35)
This relative velocity $v_T$ makes sense only for points in contact with the die and moving along the die. The Coulomb friction law can be written as:

$$t_T = -P.m - t_N n$$

$$\Phi := |t_T| - \mu t_N \leq 0$$

$$v_T = \zeta \frac{t_T}{|t_T|}$$

$$\zeta \geq 0$$

$$\zeta \Phi = 0$$

(2.36)

Following Simo and Laursen [34], we introduce the penalty parameters $\epsilon_N$ and $\epsilon_T$, and augment them with the Lagrangian variables $\lambda_N$ and $\lambda_T$ to express the inequality constraints as equality constraints. The normal traction is expressed as,

$$t_N = \langle \lambda_N + \epsilon_N g \rangle$$

(2.37)

where $\langle \cdot \rangle$ is the Macauley bracket, defined as

$$\langle x \rangle = \frac{1}{2} [x + |x|]$$

(2.38)

The frictional constraint is written as,

$$v_T - \zeta \frac{t_T}{|t_T|} = \frac{1}{\epsilon_T} \left( \frac{\bar{v}}{t_T} - \bar{\lambda}_T \right)$$

(2.39)

where, for any vector $a$ in the direction $\tau$, $\bar{a}$ is expressed as

$$\bar{a} = a^i \tau_i$$

with

$$a^i = \frac{d}{dt} a^i$$
Substituting the expression for $v_T$ from equation (2.35), equation (2.39) reduces to the following scalar equation:

$$t_T^1 - \lambda_T^1 = \epsilon_T \left( \ddot{x} - \zeta \frac{t_T^1}{\|t_T\|} \right)$$  \hspace{1cm} (2.40)$$

The above frictional constraint equation can be integrated over the time interval $[t_n, t_{n+1}]$ using Euler backward integration scheme to obtain,

$$t_{T_{n+1}}^1 - t_{T_n}^1 = \lambda_{T_{n+1}}^1 - \lambda_{T_n}^1 + \epsilon_T \left( \ddot{x}_{n+1} - \ddot{x}_n - (\zeta_{n+1} - \zeta_n) \frac{t_{T_{n+1}}^1}{t_{T_{n+1}}^1} \right)$$  \hspace{1cm} (2.41)$$

Further, we also have to satisfy the constraint equation:

$$\Phi_{n+1}^1 := \left\| t_{T_{n+1}}^1 \right\| - \mu t_{N_{n+1}} \leq 0$$  \hspace{1cm} (2.42)$$

We first introduce a trial value for $t_{T_{n+1}}^1$:

$$t_{T_{n+1}}^{1(\text{trial})} := t_{T_n}^1 + \lambda_{T_{n+1}}^1 - \lambda_{T_n}^1 + \epsilon_T (\ddot{x}_{n+1} - \ddot{x}_n)$$  \hspace{1cm} (2.43)$$

Then,

$$\Phi_{n+1}^{(\text{trial})} := \left\| t_{T_{n+1}}^{(\text{trial})} \right\| - \mu t_{N_{n+1}}$$  \hspace{1cm} (2.44)$$

If $\Phi_{n+1}^{(\text{trial})} < 0$, equation (2.36) would imply that the parameter $\zeta = 0$, and therefore, the relative velocity of the point with respect to the die is zero. In other words, the point under consideration is “sticking” to the die. We call this point as belonging to the sticking boundary. In this case, we have, $t_{T_{n+1}}^1 = t_{T_{n+1}}^{1(\text{trial})}$. If, on the other hand, $\Phi_{n+1}^{(\text{trial})} > 0$, the trial value violates the constraint equation (2.42). Therefore, the trial value must be modified so that the constraint equation is satisfied as an equality. This condition is called sliding. Thus, we will have,

$$t_{T_{n+1}}^1 = t_{T_{n+1}}^{1(\text{trial})} \quad \text{if} \quad \Phi_{n+1}^{(\text{trial})} \leq 0$$
\[ = \mu t_{n+1} \left[ \frac{1}{t_{n+1}} \right] - \frac{1}{t_{n+1}}, \quad \text{otherwise} \quad (2.45) \]

### 2.3.4 Principle of virtual work equation

Let us suppose that the configuration \( B_n \) of the body at time \( t = t_n \) is known and is under equilibrium with the forces acting on it. Under the action of the external forces, the body deforms and occupies the configuration \( B_{n+1} \) at the time \( t = t_{n+1} \).

Equation (2.5) describes the equilibrium of the body at time \( t_{n+1} \) in the updated reference configuration \( B_n \). Let \( \Gamma \) be the surface (\( \Gamma \subset \partial B_n \)) which corresponds to the regions of the body that may potentially come into contact with the die.

The incremental quasi-static boundary value problem at time \( t = t_{n+1} \) is to find the incremental (with respect to the configuration \( B_n \)) displacement field \( u(x, t_{n+1}) \equiv u_{n+1} \) that will satisfy the equation (2.5). The weak form of this equation is written as:

\[
\tilde{G}(u_{n+1}, \bar{u}) \equiv G(u_{n+1}, \bar{u}) + G_c(u_{n+1}, \bar{u}) = 0 \quad (2.46)
\]

for each test vector field \( \bar{u}(x_n) \), which is compatible with the kinematic boundary conditions, where

\[
G(u_{n+1}, \bar{u}) = \int_{B_n} P_r \frac{\partial \bar{u}}{\partial x_n} \, dV - \left( \int_{\partial B_{n+1}} \bar{t} \cdot \bar{u} \, dA + \int_{B_{n+1}} \bar{b} \cdot \bar{u} \, dV \right) \quad (2.47)
\]

and

\[
G_c(u_{n+1}, \bar{u}) = \int_{\Gamma} (t_N n \cdot \bar{u} + t_T \bar{u} \cdot \bar{u}) \, dA \quad (2.48)
\]

The equation (2.46) is a mixed form of the principle of virtual work. The internal work is expressed in the reference configuration \( B_n \) using the Piola-Kirchhoff stress,
while the external work is expressed in the current configuration where the applied surface tractions, $\mathbf{t}$, and body forces, $\mathbf{b}$, are given. The contact work is written in the reference configuration $\mathbf{B}_n$.

2.3.5 An algorithm for solving the frictional contact problem

We detail here the algorithm used to solve PVW equations incorporating contact and friction terms.

1. Initialization:

   Set $\lambda_N^{(0)} = (\lambda_N + \epsilon_N g)$ from the last time step,
   \[
   \Delta \lambda_T^{(0)} = 0,
   \]
   \[
   k = 0
   \]

2. Solve using Newton-Raphson strategy, for $\mathbf{u}^{(k)}$, the equation

   \[
   \tilde{G}(\mathbf{u}^{(k)}, \dot{\mathbf{u}}) = 0
   \]

   where $t_N$ and $t_T$ are evaluated from the equations (2.37) and (2.45) respectively.

3. Check for constraint satisfaction. If all the boundary points do remain in the “admissible region” and the boundary tractions obey the Coloumb friction law, then the iterations have converged. i.e.,

   IF $g(\mathbf{u}^{(k)}) \leq \text{TOL}_g$ AND $|t_T| \leq (1 + \text{TOL}_f) \mu (\lambda_N^{(k)} + \epsilon_N g)$ for all $\mathbf{x} \in \Gamma$
THEN converge. EXIT.

If any of the contraints are violated, augment the Lagrangian parameters, and solve the PVW equation once again. i.e.,

ELSE augment:

$$\lambda^{(k+1)}_N = \langle \lambda^{(k)}_N + \epsilon_N \gamma \rangle$$

$$t^1_{T_n}^{(trial)} = t^1_{T_n} + \Delta \lambda^1_T^{(k)} + \epsilon_T \Delta \xi$$

$$\Phi^{(trial)} = \left\| t^1_T^{(trial)} \right\| - \mu \lambda^{(k+1)}_N$$

IF $\Phi^{(trial)} \leq 0$, (sticking friction) THEN

$$\Delta \lambda^1_T^{(k+1)} = \Delta \lambda^1_T^{(k)} + \epsilon_T \Delta \xi$$

ELSE (sliding friction)

$$\Delta \lambda^1_T^{(k+1)} = \frac{t^1_{T_n}^{(trial)}}{\left\| t^1_T^{(trial)} \right\|} - \mu \lambda^{(k+1)}_N - t^1_{T_n}$$

ENDIF

$$k = k + 1$$

GOTO 2

ENDIF

### 2.3.6 Linearization of the PVW equation

In order to solve the non-linear PVW equations (2.49) for the incremental displacement field $u(x_n, t_{n+1})$, we employ a Newton-Raphson iterative scheme. This requires the linearization of the PVW equation about the unknown field $u_{n+1}$. Let
\( \mathbf{u}_{n+1}^{(k+1)} \) and \( \mathbf{u}_{n+1}^{(k)} \) be the displacement fields at the end of the \((k+1)^{th}\) step and the \(k^{th}\) step respectively during the Newton-Raphson iterative process. Then, the linearized form of equation (2.49) at any iteration step \(k\) will be,

\[
\tilde{G} \left( \mathbf{u}_{n+1}^{(k)}, \mathbf{u} \right) + \frac{\partial \tilde{G}}{\partial \mathbf{u}_{n+1}^{(k)}} \left( \mathbf{u}_{n+1}^{(k+1)} - \mathbf{u}_{n+1}^{(k)} \right) = 0 \tag{2.50}
\]

\( \tilde{G} \) consists of two terms, \( G \) and \( G_c \). In the following sections, we will provide the linearization of both terms.

**Linearization of the internal work**

The term \( G \) contains the internal work and the external work done by the body forces and surface forces not including the contact work at a die workpiece interface.

\[
dG = \int_{B_w} dP_r \cdot \frac{\partial \mathbf{u}}{\partial x_n} dV + d(\text{bodyforces + applied traction}) \tag{2.51}
\]

The linearizations for the follower forces and body forces have been dealt with in Hibbit [38]. In the problems that we deal with, we neglect the body forces. We also do not deal with explicit boundary traction distribution. The boundary conditions are usually the implicit contact conditions, or explicit displacement conditions. Therefore, here we only deal with the linearizations of the internal work term,

\[
dP_r = \det F_u T_{n+1} F_u^{-T}
\]

Complete linearization of this term is provided in Appendix C, resulting in

\[
dP_r = \det F_u \left\{ \text{tr} \left( dF_u F_u^{-1} \right) T_{n+1} - T_{n+1} \left( dF_u F_u^{-1} \right)^T \right. \\
- \text{tr} \left( \frac{1}{3\kappa} \mathbf{C} \left[ d\mathbf{E}_s \right] \right) T_{n+1} + \exp \left( \frac{p_s}{\kappa} \right) R_s^c \mathbf{C} \left[ d\mathbf{E}_s^c \right] \left( R_s^c \right)^T
\]
+ \left( dR' \mathbf{R}' \right)^T \mathbf{T}_{n+1} - \mathbf{T}_{n+1} \left( dR' \mathbf{R}' \right)^T \right) 
abla_{u}^{-T} \quad (2.52)

\mathbf{C} \text{ are the consistent tangent linearized modulii given by (see Appendix C):}

\mathbf{C} = 2\mu' \mathbf{I} + \left( \kappa - \frac{2}{3} \mu' \right) \mathbf{I} \otimes \mathbf{I} + \beta \mathbf{T}'_s \otimes \mathbf{T}'_s \quad (2.53)

\text{with}

\mu' = \eta_{n+1} \mu

and

\beta = \frac{3\mu_c}{\sigma^2} (c - \eta_{n+1}), \quad c = \frac{b_2}{a_1 b_2 + a_2 b_1}

a_1 \equiv 1 + 3\mu \Delta t \frac{\partial f}{\partial \sigma_{n+1}}, \quad a_2 \equiv 3\mu \Delta t \frac{\partial f}{\partial s_{n+1}}

b_1 \equiv \Delta t \frac{\partial g}{\partial \sigma_{n+1}}, \quad b_2 \equiv 1 - \Delta t \frac{\partial g}{\partial s_{n+1}}

\text{The linearization of the contact stiffness}

The linearization of the contact work \( G_c \) results in

\[ dG_c = \int_{\Gamma} \left( d(t_N \mathbf{n}) \cdot \mathbf{u} + dT \cdot \mathbf{u} \right) \, dA \]

Differentiating the equations (2.37) and (2.45), we can shown that

\[ \int_{\Gamma} d(t_N \mathbf{n}) \cdot \mathbf{u} \, dA = \int_{\Gamma} \left( H(t_N) \epsilon_N(\mathbf{n}, d\mathbf{u}) + \frac{t_N \kappa}{1 + g\kappa} (\mathbf{r} \cdot d\mathbf{u})(\mathbf{r} \cdot \mathbf{u}) \right) \, dA \quad (2.54) \]

and,

\[ \int_{\Gamma} dT \cdot \mathbf{u} \, dA = \int_{\Gamma} \left( \frac{\dot{T}^I}{\|r\| (1 + g\kappa)} (\mathbf{r} \cdot d\mathbf{u}) \left( y \cdot \xi \cdot \mathbf{u} \right) + \frac{\dot{T}^I}{\|r\|} r \cdot \mathbf{u} \right) \, dA \quad (2.55) \]
If the boundary is sticking \((i.e., \Phi < 0)\), then

\[
\frac{d}{dt} T \cdot \ddot{u} = \frac{c_T}{1 + g \kappa} (r \cdot d\mathbf{u}) (r \cdot \ddot{u})
\]  

(2.56)

If sliding, then

\[
\frac{d}{dt} T \cdot \ddot{u} = -\mu \text{sign}(t_T)(n \cdot d\mathbf{u}) (r \cdot \ddot{u}) - \frac{t_T}{|T|} \left( \frac{y_{,x} \cdot \ddot{u}}{n \cdot d\mathbf{u}} \right) (r \cdot d\mathbf{u}) (r \cdot \ddot{u})
\]

(2.57)

where, in the above, \(H(\cdot)\) is the heavyside function, equal to 1 only when the argument is positive, and 0 otherwise. Also, the curvature \(\kappa\) is given by,

\[
\kappa = \frac{-n \cdot y_{,x}}{|T|^2}
\]

(2.58)

### 2.4 Finite element implementation

A finite element code was written in order to implement the algorithm described in the previous sections. Both axially symmetric and plane strain problems can be solved using this code. The domain of the problem is discretized by a finite element mesh and the displacement fields are expressed by means of the finite element shape functions. For all the examples worked out, cross-triangular element configurations were used in the finite element discretizations as they inherently maintained the volume constancy. However, any other element combination can be used, provided reduced integration is performed for the pressure terms during the calculation of the stiffness matrix. In problems involving curved dies, quadratic elements were employed so that the elements at the boundary could better match the curved die surface.
The finite element analysis involves obtaining the incremental displacements for each time increment by solving the equilibrium equations and the constitutive equations. Newton-Raphson linearizations are employed to solve the non-linear systems of equations.

2.4.1 The convergence criteria

The Newton-Raphson iterations for each time step are continued until a convergence is achieved on the solutions. We use three different convergence criteria to monitor this convergence. During each time increment, the incremental plastic work,

\[ \Delta W^p = \int_{t_n}^{t_{n+1}} \int_B \sigma \dot{e} \, dV \, dt \]

is used as one of the criteria for checking the convergence. The Newton-Raphson iterations are assumed to have converged when the incremental plastic work in two successive iterations fall within a suitable tolerance (0.001\%). Convergence is also monitored using the residual force terms, \( R_f \), and the incremental displacement, \( R_d \), in each of the Newton-Raphson iterations. Let \( N \) is the number of nodal unknowns in the problem, \( [K_{ij}] \) is the matrix of assembled stiffness, \( f_i \), the residual forces, and \( d_i \), the incremental displacements that need to be computed. Then,

\[
\sum_{j=0}^{N} K_{ij} d_j = f_i, \quad \text{for each } i
\]

(2.59)

Then, we define,

\[
R_f = \frac{1}{N} \sqrt{\sum_{i=0}^{N} (f_i)^2}
\]

\[
R_d = \frac{1}{N} \sqrt{\sum_{i=0}^{N} (d_i)^2}
\]

(2.60)
The convergence criteria are $R_f < 10^{-5}$ and $R_d < 10^{-8}$. The tolerances have to be chosen keeping in mind the units that are employed for the tractions and displacements.

### 2.4.2 An automatic time stepping algorithm

An automatic time stepping algorithm discussed in [28] is employed so that the incremental plastic strain in every time increment is kept close to a prescribed fixed value. Let $\dot{\varepsilon}_p$ be the prescribed value and $\dot{\varepsilon}_{p,\text{max}}$ the maximum attainable effective plastic strain increment over the previous time step. Then the factor $R$ is given by

$$R = \frac{\dot{\varepsilon}_{p,\text{max}}}{\dot{\varepsilon}_p}$$

If $R > 1.25$, then the solution is rejected, a new time increment which is smaller by a factor of $(0.85/R)$ is taken and the whole step is repeated. If $R \leq 1.25$, then the solution is accepted and the time increment for the subsequent step is determined so that the ratio $R$ is made close to 1.0. The following algorithm is used.

If $0.8 < R \leq 1.25$ then $\Delta t_{n+1} = \Delta t_n / R$

If $0.5 < R \leq 0.80$ then $\Delta t_{n+1} = 1.25 \Delta t_n$

If $R \leq 0.50$ then $\Delta t_{n+1} = 1.50 \Delta t_n$

A value of 0.02, as recommended in [28] is employed for $\dot{\varepsilon}_{p,\text{max}}$ in all the example problems.
2.4.3 The finite element algorithm

A brief sketch of the algorithm for the finite element program is given below. For a generic \((n + 1)^{th}\) step,

1. Initialize the \((n + 1)^{th}\) configuration to that of \(n^{th}\) configuration. Start the Newton-Raphson iteration:

2. Obtain the stiffness and the unbalanced forces element by element, and assemble them to get the global Jacobian and unbalanced load vector for the \(k^{th}\) step.

   (a) Evaluate the deformation gradient at each gauss point on a given element.

   (b) Evaluate the consistent tangent moduli.

   (c) Evaluate the element stiffness.

   (d) Evaluate the load correction from the previous iteration from the Piola-Kirchoff stress.

   (e) Evaluate the stiffness and the force term due to the boundary tractions and the body forces.

   (f) Evaluate the contribution from the contact and friction to stiffness and force.

   (g) Assemble the global stiffness matrix and force vector.

   (h) Impose the displacement boundary constraints.

3. Solve for the incremental displacements.
4. Check the convergence criteria. If not converged, repeat the steps from 2 to
4. If converged, go to step 5.

5. Compute the stresses and the state variables.

6. Update the current configuration and go to step 1.

2.5 Numerical examples

2.5.1 The constitutive model

The sine hyperbolic model for Al 1100-O at 673 K provided in [29] is given below.

This is the material model used in all the examples.

The flow rule is given by

\[ f(\bar{\sigma}, s) = A \left( \sinh \left( \frac{\bar{\sigma}}{s} \right) \right)^{1/m} \]  \hspace{1cm} (2.61)

and the hardening function is given by

\[ g(\bar{\sigma}, s) = h(\bar{\sigma}, s) \cdot f(\bar{\sigma}, s) \]  \hspace{1cm} (2.62)

where the function \( h \) is given by

\[ h(\bar{\sigma}, s) = h_0 \left| 1 - \frac{s}{s^*} \right|^n \]  \hspace{1cm} (2.63)

and

\[ s^* = \bar{s} \left( \frac{\tilde{\varepsilon}^p}{A} \right)^n \]  \hspace{1cm} (2.64)

The specific values of the material parameters are given in Table (2.1).
Table 2.1: Material parameters for Al 1100-O at 673 K

<table>
<thead>
<tr>
<th>Material Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$4.75 \times 10^{-7}$ sec$^{-1}$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>7.0</td>
</tr>
<tr>
<td>$m$</td>
<td>0.23348</td>
</tr>
<tr>
<td>$s_0$</td>
<td>29.7 MPa</td>
</tr>
<tr>
<td>$h_0$</td>
<td>1115.6 MPa</td>
</tr>
<tr>
<td>$a$</td>
<td>1.3</td>
</tr>
<tr>
<td>$\bar{s}$</td>
<td>18.92 MPa</td>
</tr>
<tr>
<td>$n$</td>
<td>0.07049</td>
</tr>
<tr>
<td>$\mu$</td>
<td>20.2 GPa</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>66.0 GPa</td>
</tr>
</tbody>
</table>
2.5.2 Open die forging

As a simple example of a large deformation metal forming problem, the isothermal upset forging of a cylindrical billet is considered. This is a standard benchmark problem dealt with in [28] - [31]. The cylinder has a height of 3 mm and diameter of 2 mm. The die is modeled as a rigid surface and sticking friction is assumed between the die and the work piece during contact. To apply sticking friction, a high friction coefficient of 0.8 was used at the die-workpiece interface. The deformation is highly non-homogeneous. The symmetry of the problem enabled modeling of just one fourth of the geometry. A nominal strain rate of 0.01 was applied during the forging. 320 six noded cross-triangular finite elements were used to discretize the domain.

The billet was subjected to a height reduction of 66.67%. Automatic time stepping algorithm was used so that the maximum plastic strain increment in a step did not go beyond a limiting value of 0.02. The finite element mesh at the beginning and the end of the deformation is shown in Figure (2.4). Considerable fold over occurs at the corner near the die-workpiece interface with 9 new nodes coming into contact with the die. The contour plots of the equivalent plastic strain and the internal state variable are shown in Figures (2.5) and (2.6) respectively. The die force versus the stroke is plotted in Figure (2.7).
Figure 2.4: Finite element mesh for an axisymmetric upsetting problem. Deformed mesh is shown after 66.67% upsetting.

Figure 2.5: Contours of the effective plastic strain $\tilde{\varepsilon}^p$.

Figure 2.6: Contours of the internal state variable $s$. 
Another important metal forming problem considered is an axisymmetric extrusion of a workpiece through a die, with the reduction in area of 50%. A similar problem was considered in [34] but with different boundary conditions, as the problem was modeled as drawing. We use the extrusion boundary conditions here as they are physically more meaningful for the initial workpiece considered. The die is modeled as a quintic polynomial which has zero slope and zero curvature at the inlet and the outlet. The workpiece is pushed through the die at a constant strain rate of 0.01 sec\(^{-1}\). A friction coefficient of 0.1 is assumed at the die-workpiece interface. 144 six noded cross-triangular elements are used to discretize the geometry. Earlier experiments with using linear elements did not yield desired results as the boundary of the mesh could not follow the curved die shape at the exit with the contact.
Figure 2.8: Finite element mesh for an axisymmetric extrusion, where the cross-sectional area is reduced by 50%.

Figure 2.9: Contours of effective plastic strain $\bar{\varepsilon}$. 

algorithm used. The linear elements artificially increased the stiffness resulting in the workpiece separating from the die.

The initial and the deformed mesh are shown in the Figure (2.8). Figures (2.9) and (2.10) show the contours of equivalent plastic strain and the internal state variable respectively.

### 2.5.4 Plane strain frictionless forging

A plane strain forging problem, where a cylindrical aluminum billet is converted from a circular cross section to a crucible shape, presented in [28] is solved here. This problem presents several interesting features such as a high degree of non-homogeneous deformation, variable regions of contact between the billet and dies,
time varying deformation rates at material points and a rapid rise of the total die force when the dies become filled.

The die-workpiece is considered frictionless. The symmetry of the workpiece is used to model only one quarter of the workpiece. 432 six noded cross-triangular elements are used to discretize the geometry. The deformed mesh is shown in the Figure (2.11). Figures (2.12) and (2.13) show the contours of the equivalent plastic strain and the internal state variable respectively. Die force vs the stroke is shown in Figure (2.14). As can be noted from the die forces, there is a sudden increase in the forging forces required as the workpiece is squeezed to fill the dies.

2.5.5 Closed die forging

A closed die plane strain forging problem is demonstrated by forging a "U" cross section. This example demonstrates the effectiveness of the implicit friction algorithm in filling a die cavity. The workpiece is a cylinder whose volume is chosen so that it will exactly fill the die at the end of the forging process. The die-workpiece
Figure 2.11: Finite element mesh for an axisymmetric closed die forging.

Figure 2.12: Contours of effective plastic strain $\bar{\epsilon}^p$. 
Figure 2.13: Contours of internal state variable \( s \).

Figure 2.14: Die force vs Stroke for the plane strain forging.
interface is considered frictionless. 400 six noded cross-triangular elements are used
to discretize the geometry. The deformed mesh is shown in the Figure (2.15). The
material just about fills the volume of the die perfectly. Quadratic element shape
functions had to be used so that the mesh could follow the die shape without un-
reasonably increasing the number of elements. Figures (2.16) and (2.17) show the
contours of equivalent plastic strain and the internal state variable respectively. Die
force vs the stroke is shown in Figure (2.18).

2.6 Conclusions

An updated lagrangian hyperelastic viscoplastic formulation for the analysis of large
deformations problem was presented in this chapter. A radial return integration
scheme and full linearization of the principle of virtual work incorporating the con-
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Figure 2.16: Contours of effective plastic strain $\tilde{\varepsilon}$.  

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Figure 2.17: Contours of internal state variable $s$. 
tact modeling and employing the consistent material moduli was implemented. A finite element program was developed incorporating the above theory. The performance of this finite element code in solving large deformations problem was demonstrated by means of many numerical examples. This code is used in later chapters in conjunction with the sensitivity analysis for solving the die and preform design problems.

Figure 2.18: Die force vs Stroke for the closed die forging.
Chapter 3

Definition of Sensitivities

3.1 Introduction

As was seen earlier, in order to solve the optimization problems in die and preform designs using a sequential search methods, we need to obtain the sensitivities of various field quantities involved in a deformation problem. In this chapter, we define the notions of various sensitivities. We consider the following cases:

- The initial configuration of the body remains the same and perturbations are applied on the boundary conditions, or initial conditions, or material property distribution etc.

- The shape of the initial configuration itself is perturbed while maintaining every other parameter constant.

We call the sensitivities as parameter sensitivity and shape sensitivity respectively. We make this distinction because the treatment of the equilibrium equations and the
boundary conditions will vary considerably in case of the two sensitivity problems. The constitutive relations at a single point in a body however remains the same whether the sensitivities evaluated are with respect to the shape of the workpiece or the shape of the die.

Tortorelli et al. ([11] - [12]) solved a shape design problem in casting optimization by introducing a reference configuration and by mapping the various casting shapes to this reference configuration. In case of deformation problems, however, variation with respect to the shape changes have to be defined and evaluated very carefully.

After defining the sensitivities, we develop sensitivity constitutive equations. Then, we propose an Euler backward integration scheme to solve for the sensitivity fields of the stresses and the state variables, given the history of the variation in the deformation gradient. Then, we present a simple case study involving homogeneous deformation to evaluate the robustness of the integration scheme presented here at a single point.

\section*{3.2 Definitions of sensitivities}

\subsection*{3.2.1 Reference configuration}

In a typical Lagrangian analysis of a deformation process, material occupying an initial configuration \((t = 0)\) is deformed to obtain a final configuration. Since we are interested in the variation of Lagrangian fields induced by a variation in the initial shape of the body, we need to define an imaginary, \textit{fixed}, reference configuration. This reference configuration, under appropriate smooth geometric mappings results
in different initial configurations. The choice of the reference configuration can be completely arbitrary.

We consider the body in the following configurations (see Figure 3.1):

- the *reference* configuration: \( \mathbf{B}_R \).

  \( \mathbf{Y} \) will denote an arbitrary point in this configuration. The points of this configuration do not have any material significance.

- the *material* configuration: \( \mathbf{B}_0 \), at time \( t = 0 \).

  This configuration represents the initial state of the material that undergoes the deformation process, and the material points of this configuration will be denoted by \( \mathbf{X} \). The configuration \( \mathbf{B}_0 \) is completely described by a sufficiently smooth *reference map* of the form:

\[
\mathbf{X} = \hat{\mathbf{X}} (\mathbf{Y}; \beta_s)
\]

(3.1)
where, in general, $\beta_*$ are smooth functions that define the whole or parts of $\partial B_0$. All Lagrangian functions, $\Phi = \Phi(X, t)$, depend on the shape parameters $\beta_*$ through the variable $X$.

- the spatial configuration: $B_t$ at time $t$.

This represents the state of the body undergoing deformation after an elapsed time $t$. The points in this configurations will be denoted by $x$ and the motion of the body is described with respect to the material configuration by the equation (2.1) and with respect to the reference configuration by:

$$ x = \dot{x} \left( X(Y; \beta_*), t \right) = \dot{x}(Y, t; \beta_*) \quad (3.2) $$

All Eulerian functions, $\phi = \hat{\phi}(x, t)$, depend on the shape parameters $\beta_*$ through the variables $x$ and $X$.

**NOTE:** For a fixed reference configuration $B_R$ and a given boundary of the initial configuration defined through the functions or scalar parameters $\beta_*$, one can always construct a mapping of the form of equation (3.1). For example, some discrete mappings of this form are presented in [42].

**NOTE:** Similarly to the definitions indicated in equation (3.2), one can consider an arbitrary Lagrangian field $Z = \hat{Z}(X, t)$, and using the relation (3.1) write this function as,

$$ Z = \hat{Z} \left( \dot{X}(Y; \beta_*), t \right) = \hat{Z}(Y, t; \beta_*) $$

Similarly, we write an Eulerian variable field, $z = \hat{z}(x, t)$ as

$$ \hat{z}(x, t) = \hat{z} \left( \dot{x} \left( \dot{X}(Y; \beta_*), t \right), t \right) = \hat{z}(Y, t; \beta_*) $$
3.2.2 Definition of the parameter sensitivities

Let \( Z(X, t; \beta_p) \) be a Lagrangian field for a particular value of the parameter \( \beta_p \), which can be a scalar, vector or a tensor function. For example, in a metal forming process, \( \beta_p \) may represent the die surface, where, each die surface \( \beta_p \) defines a particular set of boundary conditions. Or, \( \beta_p \) could be the initial material property distribution over the entire domain. The field \( Z \) will be different, if the value of the parameter \( \beta_p \) changes.

The \textit{parameter sensitivity} \( \hat{Z}(X, t; \beta_p, \Delta \beta_p) \) of \( Z(X, t; \beta_p) \) is the total Gateux directional derivative of \( Z \) in the direction of \( \Delta \beta_p \) computed at \( beta_p \) [45]. \( i.e., \)

\[
\frac{d}{d\lambda} Z(X, t; \beta_p + \lambda \Delta \beta_p) \bigg|_{\lambda=0}
\]

This parameter derivative or sensitivity can be understood as the difference between two fields representing the same physical quantity \( Z \), resulting due to two different process parameters \( \beta_p \) and \( \beta_p + \Delta \beta_p \), \( i.e., \) (refer to Figure (3.2))

\[
\hat{Z}(X, t; \beta_p, \Delta \beta_p) = Z(X, t; \beta_p + \Delta \beta_p) - Z(X, t; \beta_p) + \mathcal{O}(\| \Delta \beta_p \|^2)
\]

\textbf{NOTE:} The quantity \( \hat{Z} \) is of the same nature as \( Z \) (ie., if \( Z \) is a second order tensor, then \( \hat{Z} \) also is a second order tensor). Note that the Fréchet derivative is a much stronger definition of the differential [45]. If the Fréchet derivative exists at a given point, then the Gateux differential also exists at the same point. However, the existence of the Gateux differential does not imply the existence of the Fréchet derivative. If the Fréchet derivative (gradient) exists at a point, one can construct
an adjoint problem to determine this gradient from the Gateaux differential (for examples in the area of heat transfer see [22] and [21]). Such adjoint infinite dimensional formulations have not yet been investigated for large deformation problems mainly because of the complicated mathematical nature of the governing differential equations. Note that in the sensitivity analysis literature, the difference between the Fréchet and Gateux differentials has not been strongly emphasized mostly because one is dealing with process parameters that belong to finite dimensional spaces.

Following the previous definition of sensitivity, we have,

$$
\left( \frac{\partial Z}{\partial t} \right)^* = \frac{\partial}{\partial t} \left( \frac{\hat{Z}}{Z} \right)
$$

(3.4)
and
\[
\left( \frac{\partial \hat{Z}}{\partial X_i} \right)^* = \frac{\partial}{\partial X_i} \left( \hat{Z} \right)
\] (3.5)

where \( X_i \) are the components of the position vector \( \mathbf{X} \) in the reference configuration.

Based on the definitions of the sensitivity of a Lagrangian field, one can now define the sensitivity \( \hat{Z} (\mathbf{x}, t; \beta_p, \Delta \beta_p) \) of an Eulerian field \( \mathbf{z}(\mathbf{x}, t; \beta_p) \). Examples of such Eulerian fields are the Cauchy stress \( \mathbf{T} \), the plastic work \( w^p \), the scalar variable \( s \) and the plastic strain rate \( \dot{\varepsilon}^p \). From the relationship between the current position coordinates and the reference position as expressed in equation (2.1), one can determine the sensitivity of an Eulerian field as follows:

\[
\hat{z} (\mathbf{x}, t; \beta_p, \Delta \beta_p) = \frac{d}{d\lambda} \mathbf{z}(\mathbf{x}, t; \beta_p + \lambda \Delta \beta_p) \bigg|_{\lambda=0}
\]

\[
= \frac{d}{d\lambda} \mathbf{z}(\dot{\mathbf{x}} (\mathbf{X}, t; \beta_p + \lambda \Delta \beta_p), t; \beta_p + \lambda \Delta \beta_p) \bigg|_{\lambda=0}
\] (3.6)

Note that, in the above equation, \( \dot{\mathbf{x}} \) is a Lagrangian field.

### 3.2.3 Definition of the shape sensitivities

We define the shape derivative of a variable Lagrangian field \( \mathbf{Z}(\mathbf{X}, t) \) as below [44]:

The shape derivative, \( \frac{\partial}{\partial \mathbf{X}} \mathbf{Z}(\mathbf{X}, t; \beta_s, \Delta \beta_s) \) of \( \mathbf{Z}(\mathbf{Y}, t; \beta_s) \) is the total Gateaux derivative of \( \mathbf{Z} \) in the direction of \( \Delta \beta_s \) computed at \( \beta_s$:}

\[
\frac{\partial}{\partial \mathbf{X}} \mathbf{Z}(\mathbf{Y}, t; \beta_s, \Delta \beta_s) = \frac{d}{d\lambda} \mathbf{Z}(\mathbf{Y}, t; \beta_s + \lambda \Delta \beta_s) \bigg|_{\lambda=0}
\] (3.7)

The shape derivative or sensitivity can be understood as the difference between two fields representing the same physical quantity \( \mathbf{Z} \), resulting due to different
initial configurations defined by the shape parameters: $\beta_s$ and $\beta_s + \Delta \beta_s$, i.e., (refer to Figure (3.3))

$$\ddot{\hat{Z}}(\mathbf{Y}, t; \beta_s, \Delta \beta_s) = \dot{\hat{Z}}(\mathbf{Y}, t; \beta_s + \Delta \beta_s) - \dot{\hat{Z}}(\mathbf{Y}, t; \beta_s) + \mathcal{O}\left(\|\Delta \beta_s\|^2\right)$$

*It should be noted that the fields $\hat{Z}(\mathbf{Y}, t; \beta_s + \Delta \beta)$ and $\hat{Z}(\mathbf{Y}, t; \beta_s)$ are compared at points of the reference configuration $\mathbf{B}_R$. The shape derivatives provide a measure of the change in a variable field due to a small change in the initial shape of the body. The quantity $\ddot{\hat{Z}}$ is of the same nature as $\hat{Z}$. \par

Based on the definition above, the shape derivative of the time derivative of $\hat{Z}$ can be written as:

$$\frac{\partial \ddot{\hat{Z}}}{\partial t} = \frac{\partial}{\partial t} \left(\ddot{\hat{Z}}\right)$$ (3.8)
Likewise, the shape derivative of a spatial derivative of $\tilde{Z}$ with respect to the reference configuration $B_R$ can be written as:

$$\left( \frac{\partial \tilde{Z}}{\partial Y_i} \right) = \frac{\partial}{\partial Y_i} \left( \tilde{Z} \right)$$

where $Y_i$ are the components of the position $Y$ of a material point in the reference configuration.

**NOTE:** In the subsequent chapters, we adopt the notation $\nabla_X Z$ while evaluating the spatial derivatives with respect to the configuration $B_0$, and $\nabla_Y Z$, with respect to $B_R$.

The shape derivatives of various spatial derivatives with respect to the material configuration $B_0$ have to be evaluated carefully. We first introduce the *reference gradient*, $F_R$, as follows:

$$F_R = \nabla_Y X = \frac{\partial}{\partial Y} \tilde{X} (Y; \beta_s)$$

The shape derivative of the reference gradient, denoted by $\tilde{F}_R$, is defined as:

$$\tilde{F}_R = \frac{\partial}{\partial Y} \tilde{X} (Y; \beta_s)$$

We further define $L_R$ as:

$$L_R = \tilde{F}_R \ F_R^{-1}$$

The quantity $L_R$ does not depend on the configuration $B_R$. Suppose we choose another reference configuration $B'_R$, such that there is a one to one correspondence between all the points in $B_R$ and $B'_R$. Therefore, there exists a linear transformation $G$ that takes every point in the configuration $B'_R$ to $B_R$. Then, we have,

$$F'_R = F_R \ G$$
\[ F'_R = F_R G \]

and thus,
\[ L'_R = F'_R (F'_R)^{-1} = F_R F_R^{-1} = L_R \]

All spatial derivatives with respect to the configuration \( B_0 \) are expressed in terms of spatial derivatives with respect to the configuration \( B_R \) and \( F_R \) before the shape derivatives are evaluated. Consider a scalar field \( a \) and its gradient with respect to the position \( X, \nabla_X a \). By appropriate transformations, the gradient can be written as,
\[
\nabla_X a = F_R^{-T} \left[ \nabla_Y a \right]
\]

Thus,
\[
(\nabla_X a) = (F_R^{-T}) \left[ \nabla_Y a \right] + F_R^{-T} \left[ \nabla_Y \hat{a} \right]
\]

Using,
\[
(F_R^{-T}) = -F_R^{-T} \left( F_R \right)^T F_R^{-T}
\]

and transforming back, one obtains,
\[
(\nabla_X a) = \nabla_X \hat{a} - L_R^T \left[ \nabla_X a \right] \quad (3.13)
\]

Similarly, let us consider a vector field \( v \). Then, \( \nabla_X v \) is written as,
\[
\nabla_X v = \nabla_Y v \ F_R^{-1}
\]

Therefore,
\[
(\nabla_X v) = \nabla_X \hat{v} - \nabla_X v \ L_R \quad (3.14)
\]
and
\[
\left(\nabla_X \cdot \mathbf{v}\right) = \nabla_X \cdot \mathbf{\hat{v}} - (\nabla_X \mathbf{v}) \cdot \mathbf{L}_R^T
\]
Finally, let us consider a tensor field \( A \) and its divergence with respect to the configuration \( B_0, \nabla_X \cdot A \). This can be written as,
\[
\nabla_X \cdot A = \nabla_Y A \left[ F_R^{-T} \right]
\]
The shape derivative of the quantity \( \nabla_X \cdot A \), can be calculated as follows:
\[
\left(\nabla_X \cdot A\right) = \nabla_X \cdot \mathbf{\hat{A}} - \nabla_X A \left[ L_R^T \right]
\]
or upon further simplification,
\[
\left(\nabla_X \cdot A\right) = \nabla_X \cdot \left( \mathbf{\hat{A}} - A \mathbf{L}_R \right) + A \left[ \nabla_X \cdot L_R^T \right]
\]
\[\text{(3.15)}\]
\[\text{(3.16)}\]

**NOTE:** We can now obtain the shape derivatives of time and spatial derivatives of Eulerian fields. We first express all the Eulerian fields in the reference configuration \( B_R \), then perform the shape differentiation and finally express them in the current configuration. This is possible in a Lagrangian framework, while there may be some difficulties in an Eulerian analysis.

### 3.3 The sensitivity deformation problem

In the following sections, we assume that the material state and deformation history for each parameter \( \beta \) (either \( \beta_p \) or \( \beta_s \)) are known from the solution to the corresponding direct problem described in the previous chapter. As such, the dependence of the sensitivity fields on the history of material state and deformation will not be
shown explicitly. Whenever no subscripts are explicitly stated, the variable fields of the direct problem are assumed to be evaluated at time $t = t_{n+1}$.

We classify the sensitivity problem into two subproblems, the constitutive problem and the kinematic problem. As far as the constitutive problem is concerned, the nature of the sensitivity (shape or parameter) does not matter. We will therefore use the notation that we have been using for the shape sensitivities.

In the constitutive problem, it is assumed that the sensitivities of the triad $V_n^\circ = (T_n, F_n^p, s_n)$ are known at time $t = t_n$. The objective is to determine $V_{n+1}^\circ$, from knowing the sensitivities of the total deformation gradient $\tilde{F}_{n+1}^\circ$.

In the kinematic problem, using a principle of virtual work like equation derived from the sensitivities of the equilibrium equation, one has to evaluate the sensitivities of the deformation gradient $\tilde{F}_{n+1}^\circ$ at time $t = t_{n+1}$ knowing $V_n^\circ$, the linear relationship between $V_{n+1}^\circ$ and $\tilde{F}_{n+1}^\circ$, and by applying appropriate boundary conditions for $\tilde{F}_{n+1}^\circ$. The shape derivative of the equilibrium equation will be considerably different from the parameter derivative of the same. We will describe these equations in the subsequent chapters entirely devoted to each.

The kinematic and the constitutive problems are coupled together and result in a single linear system of equations if a finite element method is employed to solve this initial and boundary value problem. In this chapter we will deal only with obtaining the relationship between various sensitivity fields involved in the constitutive equations.
3.3.1 Shape derivatives of the constitutive equations

In solving the direct constitutive problem, it is customary to employ a radial return method ([29] and [32]) and introduce the concept of a “trial stress.” However, the sensitivity constitutive problem is linear and as will be shown later, an Euler backward integration scheme can be carried out explicitly to determine the relationship between the sensitivities of the material state and the sensitivities of the deformation gradient.

Let us first start with finding the sensitivity of the Cauchy stress $\mathbf{T}$. Linearizations similar to the ones performed in the direct analysis as part of the calculation of the tangent stiffness and of the material consistent moduli, are used here to calculate the sensitivities of the material state. Following Appendix C and [39], [32], one can express $\mathbf{T}$ as,

$$\mathbf{T} = (\det \mathbf{U}^\varepsilon)^{-1} \mathbf{R}^\varepsilon \mathbf{\mathbf{T}}^\varepsilon (\mathbf{R}^\varepsilon)^T - \text{tr} \left( \mathbf{\mathbf{E}}^\varepsilon \right) \mathbf{T} + 2 \text{sym} \left( \mathbf{\mathbf{R}}^\varepsilon (\mathbf{R}^\varepsilon)^T \mathbf{T} \right)$$

(3.17)

where,

$$\mathbf{\mathbf{R}}^\varepsilon (\mathbf{R}^\varepsilon)^T = \mathbf{\mathbf{F}}^\varepsilon (\mathbf{F}^\varepsilon)^{-1} - \mathbf{R}^\varepsilon \mathbf{U}^\varepsilon (\mathbf{U}^\varepsilon)^{-1} (\mathbf{R}^\varepsilon)^T$$

(3.18)

and

$$\mathbf{\mathbf{U}}^\varepsilon = \text{sym} \left( (\mathbf{U}^\varepsilon)^{-1} \text{sym} \left( (\mathbf{F}^\varepsilon)^T \mathbf{\mathbf{F}}^\varepsilon \right) \right)$$

(3.19)

By employing the First Padé approximation for the logarithm of $\mathbf{U}^\varepsilon$, and taking the shape derivatives, $\mathbf{\mathbf{E}}^\varepsilon$ can be written as:

$$\mathbf{\mathbf{E}}^\varepsilon = 4 (\mathbf{U}^\varepsilon + \mathbf{I})^{-1} \mathbf{U}^\varepsilon (\mathbf{U}^\varepsilon + \mathbf{I})^{-1}$$

(3.20)
\( \mathbf{T} \) is expressed as:

\[
\mathbf{T} = L' \left[ \mathbf{E}' \right]
\]

By following the definitions of \( \mathbf{T}' \) and \( \bar{\sigma} \), the shape derivatives, \( \mathbf{T}' \) and \( \bar{\sigma} \) are obtained as:

\[
\mathbf{T}' = \mathbf{T} - \frac{1}{3} \text{tr} \left[ \mathbf{T} \right] \mathbf{I}
\]

and

\[
\bar{\sigma} = \frac{3 \mathbf{T}' \cdot \mathbf{T}'}{2 \bar{\sigma}}
\]

The relation between any two of the sensitivities mentioned above is completely linear. As such, substitution of equations (3.18) - (3.23) into equation (3.17) results in a linear relation between \( \mathbf{T} \) and \( \mathbf{F}^e \).

After determining the linear relation between \( \mathbf{T} \) and \( \mathbf{F}^e \), we now have to determine the relation between \( \mathbf{F}^e \) and \( \mathbf{F} \) at the end of each time step. Starting from the multiplicative decomposition of the deformation gradient, we obtain,

\[
\mathbf{F} = \mathbf{F}^e \mathbf{F}^p + \mathbf{F}^e \mathbf{F}^p
\]

Hence,

\[
(F^e)^{-1} \left( \mathbf{F} \mathbf{F}^{-1} \right) F^e = (F^e)^{-1} \mathbf{F}^e + \mathbf{F}^p \left( \mathbf{F}^p \right)^{-1}
\]

If we can determine \( \mathbf{F}^p \left( \mathbf{F}^p \right)^{-1} \), then we can solve the equation (3.24) to obtain the relation between \( \mathbf{F}^e \) and \( \mathbf{F} \). Consider the following:

\[
\frac{d}{dt} \left( \mathbf{F}^p \left( \mathbf{F}^p \right)^{-1} \right) = \frac{d}{dt} \left( \mathbf{F}^p \right) \left( \mathbf{F}^p \right)^{-1} - \mathbf{F}^p \left( \mathbf{F}^p \right)^{-1} \frac{d}{dt} \left( \mathbf{F}^p \right) \left( \mathbf{F}^p \right)^{-1}
\]
Using the equations (3.8) and the evolution equation for $\bar{F}^p$, we obtain,

$$\frac{d}{dt} \left( \bar{F}^p (\bar{F}^p)^{-1} \right) - \bar{D}^p \bar{F}^p (\bar{F}^p)^{-1} + \bar{F}^p (\bar{F}^p)^{-1} \bar{D}^p = \bar{D}^p$$  \hspace{1cm} (3.25)

We employ an Euler backward integration scheme over $(t_n, t_{n+1})$ to integrate the equation (3.25) and write that as:

$$\frac{d}{dt} \left( \bar{F}^p (\bar{F}^p)^{-1} \right) - \bar{D}_{n+1}^p \bar{F}^p (\bar{F}^p)^{-1} + \bar{F}^p (\bar{F}^p)^{-1} \bar{D}_{n+1}^p = \bar{D}_{n+1}^p$$

Premultiplying by $\exp(-t \bar{D}_{n+1}^p)$ and postmultiplying by $\exp(t \bar{D}_{n+1}^p)$ and integrating, we obtain,

$$\bar{F}_{n+1}^p \left( \bar{F}_{n+1}^p \right)^{-1} = \Delta \bar{F}^p \bar{F}_n^p \left( \bar{F}_n^p \right)^{-1} \left( \Delta \bar{F}^p \right)^{-1} + \Delta t \bar{D}_{n+1}^p$$ \hspace{1cm} (3.26)

where, from the direct problem,

$$\Delta \bar{F}^p = \bar{F}_{n+1}^p \left( \bar{F}_{n+1}^p \right)^{-1} = \exp \left( \Delta t \bar{D}_{n+1}^p \right)$$ \hspace{1cm} (3.27)

and $\Delta t = t_{n+1} - t_n$.

The sensitivity of $\bar{D}^p$ results in,

$$\bar{D}^p = \frac{3}{2} \frac{f(\tilde{\sigma}, \bar{s})}{\sigma} \tilde{T} + \left( \frac{\sigma f_{\tilde{\sigma}} - f}{\sigma^2} + \frac{f_{\bar{s}}}{\bar{s}} \right) \tilde{T}'$$ \hspace{1cm} (3.28)

For the calculation of $\bar{s}$, we resort to the shape derivative of the evolution equation for $s$. From this, the evolution equation for $\bar{s}$ is obtained as:

$$\frac{d}{dt} \left( \bar{s} \right) - \left( g_\bar{s} \right)_{n+1} \bar{s} = \left( g_\bar{s} \right)_{n+1} \bar{s}_{n+1}$$ \hspace{1cm} (3.29)

Euler backward integration results in,

$$\bar{s}_{n+1} = \bar{s}_n \exp \left( \Delta t \left( g_\bar{s} \right)_{n+1} \right) + \left( g_\bar{s} \right)_{n+1} \left[ \exp \left( \Delta t \left( g_\bar{s} \right)_{n+1} \right) - 1 \right] \bar{s}_{n+1}$$ \hspace{1cm} (3.30)
Substitution of equations (3.30) and (3.28) in (3.26) results in

$$\tilde{\mathbf{F}}_{n+1}^p \left( \tilde{\mathbf{F}}_{n+1}^p \right)^{-1} = C_{n+1} + a_{n+1} \tilde{T}'_{n+1} + b_{n+1} \tilde{\sigma}_{n+1} \tilde{T}'_{n+1}$$

(3.31)

where

$$C_{n+1} = \Delta \tilde{\mathbf{F}}^p \left( \tilde{\mathbf{F}}_{n+1} \left( \tilde{\mathbf{F}}_{n+1}^p \right)^{-1} \right) \left( \tilde{\mathbf{F}}^p \right)^{-1} + \frac{3 f \exp \left( \Delta t \, g_0 \right)}{2 \tilde{\sigma}} s_n \Delta t \tilde{T}'_{n+1}$$

$$a_{n+1} = \frac{3 \Delta t}{2 \tilde{\sigma}}$$

$$b_{n+1} = \frac{3}{2} \left( \frac{\sigma f \tilde{\sigma} - f}{\tilde{\sigma}^2} + \frac{f_s g_0 \left( \exp \left( \Delta t \, g_0 \right) - 1 \right)}{\tilde{\sigma} g_s} \right) \Delta t$$

(3.32)

As has already been shown, $\tilde{T}'_{n+1}$ and $\tilde{\sigma}_{n+1}$ depend only on $\tilde{\mathbf{F}}_{n+1}^e$. Therefore substitution of equation (3.31) in equation (3.24) results in a linear relation between $\tilde{\mathbf{F}}_{n+1}^e$ and $\tilde{\mathbf{F}}_{n+1}^p$.

$$\left( \mathbf{F}_{n+1}^e \right)^{-1} \left( \tilde{\mathbf{F}}_{n+1}^e \mathbf{F}_{n+1}^e \right) F_{n+1}^e - C_{n+1} =$$

$$\left( \mathbf{F}_{n+1}^e \right)^{-1} \tilde{\mathbf{F}}_{n+1}^e + a_{n+1} \tilde{T}'_{n+1} + b_{n+1} \tilde{\sigma}_{n+1} \tilde{T}'_{n+1}$$

(3.33)

We follow a sequence of steps to express $\tilde{\mathbf{F}}_{n+1}^e$ as a function of $\tilde{\mathbf{F}}_{n+1}^p$. We first define the variable $G_{n+1}$ as,

$$G_{n+1} \equiv \left( \mathbf{F}_{n+1}^e \right)^{-1} \left( \tilde{\mathbf{F}}_{n+1}^e \mathbf{F}_{n+1}^e \right) F_{n+1}^e - C_{n+1}$$

(3.34)

For an isotropic elasticity model with the Lamé parameters $\mu$ and $\lambda$,

$$\tilde{T}'_{n+1} = 2 \mu \tilde{\mathbf{E}}_{n+1}^e - \frac{2 \mu}{3} \text{tr} \left( \tilde{\mathbf{E}}_{n+1}^e \right) I$$

As shown in [32] and [29],

$$\text{tr} \left( \tilde{\mathbf{E}}_{n+1}^e \right) = \text{tr} \left( \tilde{\mathbf{U}}_{n+1}^e \left( \mathbf{U}_{n+1}^e \right)^{-1} \right) = \text{tr} \left( \tilde{\mathbf{F}}_{n+1}^e \left( \mathbf{F}_{n+1}^e \right)^{-1} \right)$$
Since \( \text{tr}(AB) = \text{tr}(BA) \), and considering the equations (3.33) and (3.34) along with the fact that the deviatoric quantities are traceless, we arrive at,

\[
\text{tr} \left( \overset{\circ}{E}^e_{n+1} \right) = \text{tr} \left( G_{n+1} \right) \quad (3.35)
\]

Next, we can eliminate \( \overset{\circ}{\sigma}_{n+1} \) from the equation (3.33) by considering the dot product of this equation with \( \overset{\circ}{T}'_{n+1} \) on either side. Consider the quantity, \( \left( (F^e)^{-1} \overset{\circ}{F}^e \right) \cdot \overset{\circ}{T}' \).

Using the shape derivative of the polar decomposition of \( F^e \), we have,

\[
\left( (F^e)^{-1} \overset{\circ}{F}^e \right) \cdot \overset{\circ}{T}' = \left( (U^e)^{-1} \left( (R^e)^T \overset{\circ}{R}^e \right) U^e \right) \cdot \overset{\circ}{T}' + \left( (U^e)^{-1} \overset{\circ}{U}^e \right) \cdot \overset{\circ}{T}'
\]

Since \( U^e \) and \( \overset{\circ}{E}^e \) commute in multiplication, \( U^e \) and \( \overset{\circ}{T} \) and hence \( U^e \) and \( \overset{\circ}{T}' \) also commute in multiplication. Thus,

\[
\left( (U^e)^{-1} \left( (R^e)^T \overset{\circ}{R}^e \right) U^e \right) \cdot \overset{\circ}{T}' = \left( (R^e)^T \overset{\circ}{R}^e \right) \cdot \left( (U^e)^{-1} \overset{\circ}{U}^e \right) \cdot \overset{\circ}{T}' = 0
\]

since \( (R^e)^T \overset{\circ}{R}^e \) is skew symmetric and \( \overset{\circ}{T}' \) is symmetric. Thus,

\[
\left( (F^e)^{-1} \overset{\circ}{F}^e \right) \cdot \overset{\circ}{T}' = \left( (U^e)^{-1} \overset{\circ}{U}^e \right) \cdot \overset{\circ}{T}'
\]

Now, from the diagonal representation of the symmetric tensor \( U^e \),

\[
U^e = QDQ^T
\]

where \( D \) is a diagonal tensor and \( Q \) is a rotation tensor. Therefore,

\[
\overset{\circ}{U}^e = Q \overset{\circ}{D} Q^T + 2 \text{sym} \left( \overset{\circ}{Q} Q^T U^e \right)
\]
Since $U^\varepsilon$ and $\tilde{T}'$ commute in multiplication and both are symmetric, $(U^\varepsilon)^{-1} \tilde{T}'$ is symmetric. Since $Q^T Q^T$ is skew symmetric and $\tilde{T}'$ symmetric, we have,

$$\left( (U^\varepsilon)^{-1} \text{sym} \left( \tilde{Q} \cdot Q^T U^\varepsilon \right) \right) \cdot \tilde{T}' = 0$$

Thus,

$$\left( (U^\varepsilon)^{-1} \tilde{U}^\varepsilon \right) \cdot \tilde{T}' = \left( Q \cdot \tilde{D} \cdot D^{-1} \cdot Q^T \right) \cdot \tilde{T}'$$

Likewise we have,

$$\tilde{E}' = \ln (U^\varepsilon) = Q \ln (D) Q^T$$

$$\tilde{E}' = Q \cdot \tilde{D} \cdot D^{-1} \cdot Q^T + 2 \text{sym} \left( Q^T \tilde{E} \cdot \tilde{E}' \right)$$

and,

$$\text{sym} \left( Q^T \tilde{E} \cdot \tilde{E}' \right) \cdot \tilde{T}' = 0$$

Thus,

$$\tilde{E}^\varepsilon \cdot \tilde{T}' = \left( Q \cdot \tilde{D} \cdot D^{-1} \cdot Q^T \right) \cdot \tilde{T}'$$

Also,

$$\tilde{\sigma} = \frac{3}{2\sigma} \tilde{T}' \cdot \tilde{T}' = \frac{3\mu}{\sigma} \tilde{E}^\varepsilon \cdot \tilde{T}'$$

Thus, we have

$$\left( (F^\varepsilon)^{-1} \tilde{F}^\varepsilon \right) \cdot \tilde{T}' = \left( (U^\varepsilon)^{-1} \tilde{U}^\varepsilon \right) \cdot \tilde{T}' = \tilde{E}^\varepsilon \cdot \tilde{T}' = \frac{\tilde{\sigma}}{3\mu} \tilde{T}'$$

Now, taking the dot product of equation (3.33) with $\tilde{T}'_{n+1}$ on either side, we obtain,

$$G_{n+1} \cdot \tilde{T}'_{n+1} = \tilde{\sigma}_{n+1} \left\{ \frac{2\tilde{\sigma} a_{n+1}}{3} + \frac{2\tilde{\sigma}^2 b_{n+1}}{3} + \frac{\tilde{\sigma}}{3\mu} \right\}$$
or,

\[ \sigma_{n+1}^o = \frac{3 \mu G_{n+1} \cdot \tilde{T}'_{n+1}}{\sigma (1 + 2 \mu (a_{n+1} + b_{n+1}))} \]

Thus, we can rewrite equation (3.33) as,

\[ H_{n+1} = \left( F_{n+1}^e \right)^{-1} F_{n+1}^e + 2 \mu a_{n+1} \bar{E}_{n+1}^e \] (3.36)

where

\[ H_{n+1} \equiv G_{n+1} + \frac{2 \mu a_{n+1} \text{tr}(G_{n+1})}{3} I - \frac{3 \mu b_{n+1}}{\sigma (1 + a_{n+1} + \sigma b_{n+1})} \tilde{T}'_{n+1} \] (3.37)

Given \( \bar{F}_{n+1}^e \) and using the definition of \( G \) from equation (3.34), we can solve the equation (3.36) numerically to obtain \( \bar{F}_{n+1}^e \). One can then write the following:

\[ \bar{F}_{n+1}^e = A \left( V_{n+1}, V_{n} \right) \left[ \bar{F}_{n+1}^e \right] + A \left( V_{n+1}, \bar{V}_{n} \right) \] (3.38)

where \( A \) is a second order tensor function and \( \bar{A} \), a fourth order tensor function. \( A \) and \( \bar{A} \) depend upon the solution \( V_{n+1} \) to the direct problem at time \( t = t_{n+1} \).

### 3.4 An example problem

To test the validity of the sensitivity constitutive equations developed in this section, we apply two known deformation gradient field histories at a single point. The sensitivity fields are obtained by using the equations developed in this section. They are compared with the finite difference solutions obtained by solving the direct constitutive relations for each of the deformational gradient histories.
In the normal cartesian coordinate system, we apply the deformation gradient histories $F$ and $F' = F + \hat{F}$,

$$F = \frac{1}{\sqrt{\lambda(t)}} e_x \otimes e_x + \lambda(t) e_y \otimes e_y + \frac{1}{\sqrt{\lambda(t)}} e_z \otimes e_z$$

$$F' = \frac{1}{\sqrt{\lambda(t)}} e_x \otimes e_x + \lambda'(t) e_y \otimes e_y + \frac{1}{\sqrt{\lambda(t)}} e_z \otimes e_z$$

where,

$$\lambda(t) = 1 + 0.01t$$

$$\lambda'(t) = 1 + (0.01 - \delta)t + \frac{t^2}{t_f}$$

where $t \in [0, t_f]$.

The above deformation gradient histories correspond to stretching a uniaxial cylindrical rod with a uniform extension rate and a non-uniform extension rate. The scalar parameter $\delta$ controlled the amount of the difference $\hat{F}$. The values of $\delta$ were varied from $10^{-2}$ to $10^{-10}$. For $\delta < 10^{-5}$, the maximum difference between finite difference fields and the direct differentiation sensitivity fields was less than 0.1%.

The comparison of the finite difference (FDM) values with the direct differentiation (DDM) values of $\hat{s}$, $\hat{\sigma}$ and the Cauchy stress component $T_{22}$, for $\delta = 10^{-6}$ are plotted in the Figures (3.4), (3.4) and (3.6) respectively.

As can be seen from the above figures, the results are extremely satisfactory. While it was possible to evaluate the finite differences for the various stress and state variable fields, in a more complicated problem such as die or preform design problems where the variable fields are computed using a finite element method,
Figure 3.4: Comparison of DDM and FDM values for $\dot{s}$

Figure 3.5: Comparison of DDM and FDM values for $\dot{\sigma}$
Figure 3.6: Comparison of DDM and FDM values for $T_{22}^*$

the accuracy of the finite difference calculations will suffer greatly and hence it is desirable to use the direct differentiation methods.
Chapter 4

Die Design Problems

4.1 Introduction

In this chapter, we develop the parameter sensitivity equilibrium equations and boundary condition for the parameter sensitivity problem. In case of the parameter sensitivity problem with changing die shapes, the driving force comes from the boundary conditions. Therefore care should be given to develop these boundary conditions. After developing the parameter sensitivity equations, the optimization method implemented for the die design problem is highlighted.

To demonstrate the effectiveness of the developed method, a die design problem is chosen in the axially-symmetric extrusion of an Aluminum workpiece. The history of the sensitivities of the stresses and the state variables are evaluated and employed in solving this optimization problem. The objective is to select a die from a given set of dies such that one achieves a distribution of the material properties in the
extruded workpiece that are as uniform as possible. For this problem, the die is represented by means of a quintic polynomial spline interpolation.

4.2 Parameter sensitivities of equilibrium equations

To solve for the sensitivity fields, one needs to consider the sensitivity of the equilibrium equation (2.5) in conjunction with a set of boundary conditions for this problem. In the following, a principle of virtual work like equation is developed for obtaining $\mathbf{F}$. The boundary conditions for this problem are treated separately in the next section.

The directional derivative of the equilibrium equations (equation 2.5) results in the following:

$$\nabla \cdot \mathbf{P}^* + \mathbf{f}^* = 0 \quad \forall \mathbf{X} \in B_0 \quad \text{and} \quad \forall t \in [0, t_f] \quad (4.1)$$

Based on the definition of $\mathbf{P}$ from equation (2.6), $\mathbf{P}^*$ is calculated as

$$\mathbf{P}^* = \det \mathbf{F} \left[ \text{tr} \left( \mathbf{F} \mathbf{F}^{-1} \right) + \mathbf{T} + \mathbf{T} - \mathbf{T} \left( \mathbf{F} \mathbf{F}^{-1} \right)^T \right] \mathbf{F}^{-T} \quad (4.2)$$

In the above equation, $\mathbf{T}$ is given by equation (3.17).

Suppose that the gravity force is the only body force acting on the body. Let $\rho_0$ be the constant density of the body in the reference configuration and let $\mathbf{g}$ be the acceleration due to gravity. The body force at any configuration is given by

$$\mathbf{b} = \rho \mathbf{g} = \frac{\rho_0}{\det \mathbf{F}} \mathbf{g}$$
hence,

\[ f = \rho_0 \ g \]

and therefore,

\[ \hat{f} = 0 \]  \hspace{1cm} (4.3)

A variational form of the sensitivity kinematic problem can now be posed as:

\[ \int_{B_0} \left( \nabla \cdot \ P \right) \cdot \dot{u}(X) \ dV = 0 \]  \hspace{1cm} (4.4)

where \( \dot{u}(X) \) is a kinematically admissible (sensitivity) field expressed over the reference configuration. At this stage, one needs to clarify certain basic assumptions regarding the space of this test displacement sensitivity field. The following discussion and analysis is restricted to the die design problem in which the reference configuration is assumed not to be affected by the various die shapes.

With integration by parts, equation (4.4) results in

\[ \int_{B_0} \ P \cdot \frac{\partial \dot{u}}{\partial X} \ dV_0 = \int_{\partial B_0} \ P \cdot m \cdot \dot{u} \ dA_0 \]  \hspace{1cm} (4.5)

where \( m \) is the unit outward normal at \( X \) in the reference configuration. Since the reference configuration is not affected by the change in the die surface, the normal to the surface in the reference configuration also is not affected. Transforming the right hand term to the current configuration (where \( n \) is the corresponding unit normal vector) and substituting the expression for \( \hat{P} \) from equation (4.2), we obtain,

\[ \int_{\partial B_0} \ P \cdot m \cdot \dot{u} \ dA_0 = \int_{\partial B_{n+1}} \left\{ \text{tr} \left( F^{-1} F^T \right) T + T - T \left( F^{-1} F^T \right)^T \right\} n \cdot \dot{u} \ dA_{n+1} \]  \hspace{1cm} (4.6)
The sensitivity of the traction vector is given as follows:

\[ \dot{t} = (T^* N) = T^* n + T^* n \]  \hspace{1cm} (4.7)

One can express the normal vector field in the current configuration \( n \) in terms of the deformation gradient field and the normal vector field in the reference configuration \( m \) as follows:

\[ n = \frac{F^{-T} m}{(F^{-T} m \cdot F^{-T} m)^{1/2}} \]  \hspace{1cm} (4.8)

Applying the sensitivity operator to the above equation and after some simplifications, one can show that:

\[ \dot{n} = - \left( F^* F^{-1} \right)^T n + \left\{ \left( F^* F^{-1} \right) \cdot n \otimes n \right\} n \]  \hspace{1cm} (4.9)

Substituting the above expression for \( \dot{n} \) in equation (4.7) and using the fact that \( \text{tr}(A) = A \cdot I \), we obtain,

\[ \dot{t} = T^* n - T \left( F^* F^{-1} \right)^T + \left( F^* F^{-1} \cdot n \otimes n \right) T n \]  \hspace{1cm} (4.10)

Thus, one finally arrives at the following weak sensitivity problem:

\[ \text{Calculate } \dot{x} (X, t) \text{ such that } \]

\[ \int_{B_0} \dot{P} \cdot \frac{\partial \mathbf{u}}{\partial X} \, dV_0 - \int_{\partial B_{n+1}} \left\{ \left( F^* F^{-1} \right) \cdot (I - n \otimes n) \right\} \dot{t} \cdot \mathbf{u} \, dA_{n+1} \]

\[ = \int_{\partial B_{n+1}} \dot{t} \cdot \dot{u} \, dA_{n+1} \]  \hspace{1cm} (4.11)

for every “admissible” test function \( \dot{u} \).

In using a finite element method to solve the above kinematic sensitivity problem, both terms in the left hand side of equation (4.11) contribute to the stiffness
matrix, while the first term also contributes to the force vector. The right hand side contributes only to the force vector. Evaluating the stiffness matrix and the force term for the above equation becomes a simple task based on the method summarized in Appendix D.

The stiffness matrix for the sensitivity problem (the left hand side of equation (4.11)) is not the same as the linearized tangent stiffness used in the direct kinematic analysis, even though the forms of \( dP \) and \( \dot{P} \) look identical. From equation (3.38), it can be seen that the relation between \( \dot{F}^s \) and \( \dot{F} \) depends on the history of the sensitivity of the state. Since the evolution of \( (T, s) \) and \( \bar{F}^p \) is not at all involved in the direct problem, the stiffness obtained from the converged direct problem cannot be used as the stiffness for the discretized sensitivity problem.

### 4.3 Initial conditions for the sensitivity problem

The following set of initial conditions are used in the direct analysis:

\[
T(X, 0) = 0 \quad \quad (4.12)
\]
\[
s(X, 0) = s_0 \quad \quad (4.13)
\]

The corresponding initial conditions for the sensitivity problem have as follows:

\[
T^*(X, 0) = 0 \quad \quad (4.14)
\]
\[
s^*(X, 0) = 0 \quad \quad (4.15)
\]
4.4 Boundary conditions for the sensitivity problem

The boundary conditions for the sensitivity problem are derived from the corresponding boundary conditions of the direct analysis. Some typical cases are examined below.

4.4.1 Free boundary

The boundary traction is zero on a free boundary. The corresponding boundary condition for the sensitivity problem can be written as follows:

\[
\mathbf{t} = 0
\]

\( (4.16) \)

4.4.2 Boundary with prescribed displacement

On this boundary, for the direct problem,

\[
x (X, t) = \hat{x} (X, t)
\]

\( (4.17) \)

where \( \hat{x} (X, t) \) is a known function. Therefore, for the sensitivity problem,

\[
\mathbf{\hat{x}}^* (X, t) = 0
\]

\( (4.18) \)
Figure 4.1: The normal and tangent vector to the die surface.

4.4.3 Boundary with mixed displacement/traction conditions

On this boundary, the component of the displacement field along one direction is known, while the traction component is prescribed in the other direction. The boundary conditions for the sensitivity problem will be similarly mixed and can be derived from the two earlier cases.

Boundary interface with a rigid die surface and sliding friction

Consider the case of plane strain or axially-symmetric problems. Let $x$ and $y$ be the position coordinates (for the case of axially symmetric problems, $x$ represents the radial direction, while $y$ is the axial direction). The die surface is represented by the equation $b(x, y) = 0$. Let $n(x, y)$ and $r(x, y)$ be the outward normal and
tangent unit vectors, respectively, at any given point \((x, y)\). Also, let \(u\), be the incremental displacement field and \(t\) the known traction field. Here, Coulomb dry friction is assumed with \(\mu\) the coefficient of friction at the die - workpiece interface. Then, the boundary conditions for the direct problem are:

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{n} &= 0 \\
\hat{t} \cdot \mathbf{r} &= \pm \mu \hat{t} \cdot \mathbf{n}
\end{align*}
\]

(4.19)

where

\[
\mathbf{u} = \mathbf{x}_{n+1} - \mathbf{x}_n
\]

The \(\pm\) sign is resolved based on the direction of the motion of the workpiece with respect to the die.

By solving the direct problem, one can obtain the traction distribution on the die surface. Let \(p(x, y)\) be the pressure distribution on the die surface (where \(p\) is a non-negative function). Then,

\[
\hat{t} = -pn \pm \mu pr
\]

(4.20)

Using the sensitivities of equations (4.19) and (4.20), one can conclude that

\[
\mathbf{u} \cdot \mathbf{n} = -\mathbf{u} \cdot \hat{\mathbf{n}}
\]

or,

\[
\mathbf{x}_{n+1} \cdot \mathbf{n} = \mathbf{x}_n \cdot \mathbf{n} - \mathbf{u} \cdot \hat{\mathbf{n}}
\]

(4.21)

and

\[
\hat{t} = p (-\mathbf{n} \pm \mu \mathbf{r}) + p \left( -\hat{\mathbf{n}} \pm \mu \hat{\mathbf{r}} \right)
\]

(4.22)
We have the following identities:

\[ n \cdot r = 0 \]
\[ n \cdot n = 1 \]
\[ r \cdot r = 1 \]  \hspace{1cm} (4.23)

The sensitivities of the above equations imply that

\[ \dot{r} \cdot n = - \dot{n} \cdot r \]
\[ \dot{n} \cdot n = 0 \]
\[ \dot{r} \cdot r = 0 \]  \hspace{1cm} (4.24)

Substituting the above equations in equation (4.22) and using the fact that for any vector \( a \) in the plane of consideration,

\[ a = (a \cdot n)n + (a \cdot r)r \]

one can obtain:

\[ \dot{t} = \left\{ \dot{p} \pm \mu p \left( \dot{n} \cdot r \right) \right\} n + \left\{ -p \left( \dot{n} \cdot r \right) \pm \mu \dot{p} \right\} r \]  \hspace{1cm} (4.25)

In summary, equations (4.21) and (4.25) constitute the set of boundary conditions for the die-workpiece interface nodes. In these boundary conditions, the change in the die surface alters only the quantity \( \dot{n} \). The calculation of \( \dot{n} \) is provided in the next section, while the implementation of the above equations is considered later in this chapter.
4.4.4 Determination of the sensitivity of the normal vector

Let \( e_x \) and \( e_y \) be the unit vectors along the \( x \) and \( y \) axes, respectively. Referring to Figure (4.1), the normal unit vector \( n(x, y) \) at any point \((x, y)\) can be written as:

\[
n(x, y) = \frac{b_x e_x + b_y e_y}{(b_x^2 + b_y^2)^{1/2}}
\]  

(4.26)

while the tangent vector is represented as:

\[
r(x, y) = \frac{-b_y e_x + b_x e_y}{(b_x^2 + b_y^2)^{1/2}}
\]  

(4.27)

where

\[
b_x = \frac{\partial b(x, y)}{\partial x}
\]

\[
b_y = \frac{\partial b(x, y)}{\partial y}
\]  

(4.28)

The equation of the die surface is written such that \((e_x, e_y, e_z)\) and \((n, r, e_z)\) form a right handed triad.

The sensitivity of equation (4.26) results in the following:

\[
n^* = \frac{(b_x b_y^* - b_y b_x^*)}{(b_x^2 + b_y^2)^{3/2}} \frac{(-b_y e_x + b_x e_y)}{(b_x^2 + b_y^2)^{3/2}}
\]

where \(b_x^*\) and \(b_y^*\) are given as

\[
b_x^* = \frac{\partial b^*}{\partial x}
\]

\[
b_y^* = \frac{\partial b^*}{\partial y}
\]  

(4.29)
Using equation (4.27), $\hat{n}$ can be written as follows:

\[
\hat{n} = \frac{\left( b_x^* b_y^* - b_y^* b_x^* \right)}{b_x^2 + b_y^2} \quad r
\]  

(4.30)

The quantity, $(\hat{n} \cdot r)$, that appears in equation (4.25), can now be obtained as follows,

\[
\hat{n} \cdot r = \frac{\left( b_x^* b_y^* - b_y^* b_x^* \right)}{b_x^2 + b_y^2}
\]  

(4.31)

Thus, the initial boundary value sensitivity problem has been defined. The entire problem has been developed in a fully infinite dimensional setting (i.e. with the die represented as a function). In subsequent sections, the implementation of this problem is discussed using the finite element method and a finite dimensional approximation of the die surface.

4.5 Finite element implementation of the die design problem

The spatial discretization of the workpiece used in solving the sensitivity problem is the same as that used in the direct analysis. Cross triangular finite elements are employed with linear shape functions. The same time step $\Delta t$ is used for the time integration of both direct and sensitivity problems. In the die design problem, several objective functionals can be considered. The gradient of these cost functionals with respect to the die surface must be evaluated. Here, a finite dimensional representation of the die surface is assumed and the gradient of a generic
field $Z(X, t; \alpha_1, \alpha_2, \ldots, \alpha_n)$ with respect to the die parameters $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is obtained from the sensitivity analysis given earlier. In the following, we first select a parametric representation for the die surface and then describe how to obtain the above gradients. Also, some details on the numerical discretization of the kinematic and constitutive sensitivity problems are provided.

### 4.5.1 Finite dimensional representation of the die surface and calculation of the gradients

Let the die surface be represented by means of a finite number of $(n + 1)$ basis functions $\phi_i(x, y)$ as

$$b(x, y; \alpha_i) = \phi_0(x, y) + \sum_{i=1}^{n} \alpha_i \phi_i(x, y)$$

(4.32)

where the $\alpha_i$'s are the $n$ scalar die parameters. Each set of these parameters corresponds to a unique die surface. Each parameter $\alpha_i$ is assumed to be bounded in such a way that a smooth family of die surfaces is obtained.

The partial derivatives of $b(x, y; \alpha_i)$ can now be written as:

$$b_x(x, y; \alpha_i) = \phi_{0x}(x, y) + \sum_{i=1}^{n} \alpha_i \phi_{ix}(x, y)$$

$$b_y(x, y; \alpha_i) = \phi_{0y}(x, y) + \sum_{i=1}^{n} \alpha_i \phi_{iy}(x, y)$$

(4.33)

where

$$\phi_{ix} = \frac{\partial \phi_i(x, y)}{\partial x}$$

$$\phi_{iy} = \frac{\partial \phi_i(x, y)}{\partial y}$$

(4.34)
Using the above finite dimensional representation of the die surface, one can approximate $b_x^*$ and $b_y^*$ as follows,

$$
\begin{align*}
  b_x^* (x, y; \alpha_i, \Delta \alpha_i) &= \sum_{i=1}^{n} \phi_{ix} \Delta \alpha_i \\
  b_y^* (x, y; \alpha_i, \Delta \alpha_i) &= \sum_{i=1}^{n} \phi_{iy} \Delta \alpha_i 
\end{align*}
$$

(4.35)

The above equations (4.33) – (4.35) can be substituted in equation (4.31) to obtain the boundary conditions for the sensitivity problem. Note that in the sensitivity problem, any arbitrary but \textit{small} values of $\Delta \alpha_i$, $i=1, 2, \ldots$ can be given as input arguments. However, the parameters $\alpha_i$, $i=1, 2, \ldots$ are uniquely defined by the current reference die surface.

Let us suppose that the sensitivity problem has been solved for a particular choice of $\Delta \alpha_i$, $i = 1, 2, \ldots, n$. Let us also consider any field $Z (X, t; \alpha_1, \alpha_2, \ldots, \alpha_n)$ whose sensitivity has already been obtained. Then, one can write:

$$
Z^* (X, t; \alpha_1, \alpha_2, \ldots, \alpha_n, \Delta \alpha_1, \Delta \alpha_2, \ldots, \Delta \alpha_n) = \sum_{i=1}^{n} \frac{\partial Z}{\partial \alpha_i} \Delta \alpha_i 
$$

(4.36)

Therefore,

$$
\frac{\partial Z}{\partial \alpha_i} = \frac{Z^* (X, t; \alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots, \Delta \alpha_i, \ldots, 0)}{\Delta \alpha_i} 
$$

(4.37)

The gradient, $\nabla Z$ of any Lagrangian field $Z$ is defined in the above parameter space as,

$$
\nabla Z = \left( \frac{\partial Z}{\partial \alpha_1}, \frac{\partial Z}{\partial \alpha_2}, \ldots, \frac{\partial Z}{\partial \alpha_n} \right)
$$

and its calculation requires the solution of $n$ sensitivity problems. In the $i^{th}$ sensitivity problem, all the incremental parameters except $\Delta \alpha_i$ are zero. The implementation of the complete die design problem involves the sequential solution at
each time step of the direct problem and the \( n \) linear sensitivity incremental problems corresponding to each of the die parameters. These \( n + 1 \) problems are solved in the whole time interval of interest and with a die reference defined by the set \( \alpha_i, i = 1, 2, \ldots, n \).

4.5.2 Calculation of the stiffness matrix for the sensitivity problem

In the solution of the sensitivity problem during the time step \( t = t_n \) to \( t = t_{n+1} \), the unknown is the sensitivity of the current position, \( \mathbf{Z}_{n+1}^* \). The variational form of the sensitivity equilibrium equation (4.1) along with the initial and the boundary conditions, results in a linear system of equation at each time step for \( \mathbf{Z}_{n+1}^* \). Using the transformations introduced in Appendix D, all terms in equation (4.1) take the form of the dot product between a known fourth order tensor and \( \mathbf{F}_{n+1}^* \) or the form of a second order tensor that does not explicitly depend upon \( \mathbf{F}_{n+1}^* \). While computationally solving problems using the finite element method, it is convenient to represent second order tensors in the form of column vectors and the fourth order tensors as square matrices. In dealing with axially symmetric and plane strain problems, every second order tensor has 5 non-zero components as shown below:

\[
X = \begin{bmatrix}
X_{11} & X_{12} & 0 \\
X_{21} & X_{22} & 0 \\
0 & 0 & X_{33}
\end{bmatrix}
\]  

(4.38)
This will be represented as:

\[
\{X_c\} = \begin{pmatrix}
X_{11} \\
X_{12} \\
X_{21} \\
X_{22} \\
X_{33}
\end{pmatrix}
\quad \text{(4.39)}
\]

We have already shown that the governing equations for the sensitivity fields are linear in nature. Various tensor operations involving an unknown second order tensor \(X\) appear in the weak form of the sensitivity problem and in other parts of the present analysis. These operations can be equivalently represented by the dot product of a proper fourth order tensor with \(X\). For example, premultiplying \(X\) by a known second order tensor \(A\) gives rise to a fourth order operator \(L_A^i\) whose components are given in appendix A. The corresponding \(5 \times 5\) matrix to this fourth order tensor has the following form:

\[
L_A^i = \begin{bmatrix}
A_{11} & 0 & A_{12} & 0 & 0 \\
0 & A_{11} & 0 & A_{12} & 0 \\
A_{21} & 0 & A_{22} & 0 & 0 \\
0 & A_{21} & 0 & A_{22} & 0 \\
0 & 0 & 0 & 0 & A_{33}
\end{bmatrix}
\quad \text{(4.40)}
\]

Thus \(B = AX\), is represented in a vector form as \(\{B_c\} = [L_A^i] \{X_c\}\).

While calculating the stiffness matrix, a representation similar to the above is employed in all the equations defined previously. For a particular selection of shape functions, one can calculate the relation between \(\hat{F}_{n+1}^*\) and \(\hat{\mathbf{x}}_{n+1}^*\) at each Gauss
point. This is exactly the same relationship as the one between the deformation gradient $\mathbf{F}_{n+1}$ and $\mathbf{x}_{n+1}$. Using this relationship and equations (4.2), (3.17) – (3.23) and (3.38), one obtains the stiffness matrix for the sensitivity problem. Finally, a linear system of equations results for the calculation of the sensitivities of the current position at each finite element node.

### 4.5.3 Application of the boundary conditions

After assembling the global stiffness matrix, the boundary conditions are imposed. Here, only the boundary conditions associated with the die-workpiece interface are discussed. For the interior nodes, the nodal quantities are evaluated along the pre-defined global axes, while for the boundary nodes the nodal quantities are resolved along the tangential and the normal directions to the die surface. Let $\hat{u}_{i_1}$ and $\hat{u}_{i_2}$ denote the sensitivities of the $i^{th}$ nodal displacements and let $\{\hat{v}\}$ denote the column vector of all (orderly numbered) nodal displacement sensitivities. For each nodal boundary point, one can calculate $\gamma = \hat{n} \cdot \mathbf{r}$ using equation (4.31).

Consider three adjacent boundary nodes that for simplicity are denoted as nodes 1, 2 and 3. We also choose a particular direction of motion for the body so that we can resolve the $\pm$ sign. Then using equation (4.25) for the sensitivity of the traction and performing the surface integration, one can write the discretized finite element equations corresponding to the boundary node 2 as follows:

\[
\sum_{q=1}^{2N} K_{pq} \hat{v}_q = W_{12}^1 \left( - \hat{p}_1 + \mu \gamma_1 p_1 \right) + \left( W_{12}^2 + W_{23}^2 \right) \left( - \hat{p}_2 + \mu \gamma_2 p_2 \right) + W_{23}^3 \left( - \hat{p}_3 + \mu \gamma_3 p_3 \right)
\]

(4.41)
\[
\sum_{q=1}^{2N} K_{(p+1)q} \, \nu_q = W_{12}^1 \left( \mu \, p^* - \gamma_1 p_1 \right) + \left( W_{12}^2 + W_{23}^2 \right) \left( \mu \, p^* - \gamma_2 p_2 \right) + W_{23}^3 \left( \mu \, p^* - \gamma_3 p_3 \right) \quad \text{(4.42)}
\]

where \( N \) denotes the total number of nodes and the index \( p \) refers to the equation number of the node 2 in the global stiffness matrix. \( W_{ij}^k \) refer to the weight due to the node \( i \) of the line segment connecting the nodes \( i \) and \( j \) while performing the surface integration on the right hand side of equation (4.1). Since we know the sensitivity displacement boundary condition along the normal direction from equation (4.21), \( p^*_i \) can be eliminated by multiplying equation (4.41) by \( \mu \) and adding to equation (4.42). After the reduced system of linear equations are solved, the \( p^*_i \) can be calculated by substituting in the above equations.

### 4.6 Numerical examples

In this section, a family of extrusion dies is considered that is generated from the interpolation of a finite number of quintic splines. From the members of this die family, an optimum die will be chosen that results in near uniform properties in the final extruded workpiece. The material considered in the following numerical example is Al 1100-O Aluminum at a temperature of 673 K. The material properties for this are given in the Table (2.1) in Chapter 2.
4.6.1 A family of dies with a quintic spline interpolation

Quintic polynomials provide a second derivative continuity over the entire die surface. Let us consider that between any two points, \( y_1 \) and \( y_2 \), the quintic polynomial is of the form:

\[
f(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5\]

The coefficients in the above polynomial are calculated uniquely by providing the value of \( f(y) \) as well as its first derivative \( f'(y) \) and its second derivative \( f''(y) \) at the two end points \( y_1 \) and \( y_2 \).

In this work, we consider \( n \) splines that connect the entrance and exit locations to the deformation zone and pass through \( (n - 1) \) intermediate points. The triad \((f(y_i), f'(y_i), f''(y_i)), i = 1, 2, \ldots, (n + 1)\) is considered to define the patch of the \( n \) splines so that up to second derivative continuity is achieved.

The initial die design for axisymmetric extrusion

An initial die shape is considered with 6 variable parameters. The die is assumed to be a curved region where the reduction takes place. Two straight regions are considered on each side of the die to guide the motion of the workpiece towards and out of the die, respectively. The length of the straight edged region near the entrance is denoted by \( c \) whereas at the exit it is denoted by \( d \) (refer to Figure (4.2) for details). The length over which the reduction takes place is \( l \). The value of the radius as well as the first and second derivatives are assumed to be known at the beginning \(((y-c)/l = 0)\) and end \(((y-c)/l = 1)\) of the deformation zone. In addition
to the end points, two intermediate points, \((y - c)/l = 1/3\) and \((y - c)/l = 2/3\), are considered. The values of the die radius and the first and second derivatives at these intermediate points are used to define the various members of the die family. The coefficients of the equation of the die surface must be determined separately in each of the three segments. The die equation is represented as,

\[ x = f((y - c)/l) \]

The required data for obtaining the unknown coefficients are,

\[ f(0) = C_1 \]
\[ f'(0) = C_2 \]
\[ f''(0) = C_3 \]
\[ f(1) = C_4 \]
\[ f'(1) = C_5 \]
\[ f''(1) = C_6 \]

The unknown die parameters are here chosen as:

\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6
\end{pmatrix} =
\begin{pmatrix}
f(1/3) \\
f'(1/3) \\
f''(1/3) \\
f(2/3) \\
f'(2/3) \\
f''(2/3)
\end{pmatrix}
\]  
(4.43)
The corresponding basis functions are provided in Appendix E.

A cylindrical workpiece of radius 1 cm is reduced by 25% at a constant strain rate of $0.01 \text{sec}^{-1}$. A constant time step $\Delta t$ of 10 secs is employed. The initial die design is chosen so that it is as close as possible to a straight die, while satisfying the required end conditions. The selected initial die parameters are given as follows:

$$
\begin{pmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4 \\
C_5 \\
C_6
\end{pmatrix} = 
\begin{pmatrix}
1.0 \\
0.0 \\
0.0 \\
\sqrt{3} \\
0.0 \\
0.0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6
\end{pmatrix} = 
\begin{pmatrix}
0.9603411660 \\
-0.081987525 \\
0.0 \\
0.90568331 \\
-0.081987525 \\
0.0
\end{pmatrix}
$$

In the examples considered, the following parameters were selected:

$$
c = 0.5 \text{ cm}$$

$$
l = 2.0 \text{ cm}$$

$$
d = 0.5 \text{ cm}$$

(4.44)
The domain is discretized by 240 three noded cross triangular element. Sliding friction with a coefficient of friction 0.005 is assumed to exist in the die-workpiece interface. The process is assumed to be completed when a fixed volume of material (4.338 cubic cm) is extruded out of the die (time \( t_f \)). In the following discussion, exit cross section refers to the cross section of the workpiece at \( y = c + l + d \) and at time \( t_f \).

Direct simulation results are obtained for the initial die design and are presented below. The undeformed and the deformed meshes for the above simulation are shown in Figure 4.2. The stresses and the state variable distributions are calculated throughout the domain. Figure 4.3 shows the contour plot of the scalar state variable (internal resistance to plastic flow) \( s \) at the end of the process (time \( t_f \) defined as part of the analysis), while Figure 4.5 presents the contour plot of the equivalent stress \( \tilde{\sigma} \). The distribution of the state variable \( s \) is plotted at the exit cross section in Figure 4.4, while the distribution of \( \tilde{\sigma} \) is given in Figure 4.6.

**Calculation of sensitivities**

In order to show the effectiveness of the sensitivity analysis developed in this work, the sensitivities of various distributions at the exit cross section for the extrusion problem are calculated and compared with the ones obtained through the finite differences method.

To obtain the sensitivities by finite differences methods for the initial die, each of the six parameters \( \alpha_i \) in the initial die design were varied one at a time by a small amount \((1 \times 10^{-4})\) and direct simulations were performed for the corresponding dies.
Figure 4.2: Deformed and undeformed meshes for the axisymmetric extrusion problem.

Figure 4.3: Contour plot of the state variable $s$. 

\begin{itemize}
    \item $1 - 29.85$ MPa
    \item $2 - 30.46$ MPa
    \item $3 - 31.07$ MPa
    \item $4 - 31.67$ MPa
    \item $5 - 32.28$ MPa
    \item $6 - 32.89$ MPa
    \item $7 - 33.50$ MPa
    \item $8 - 34.11$ MPa
\end{itemize}
Figure 4.4: Distribution of $s$ at the exit.

Figure 4.5: Contour plot of the effective stress $\sigma$. 

$1 - 5.07$ MPa  
$2 - 6.23$ MPa  
$3 - 7.39$ MPa  
$4 - 8.55$ MPa  
$5 - 9.71$ MPa  
$6 - 10.87$ MPa  
$7 - 12.03$ MPa  
$8 - 13.19$ MPa
Figure 4.6: Distribution of $\bar{\sigma}$ at the exit.

Note that all six parameters $\alpha_i$ are approximately of the same order of magnitude. At the exit cross section, the distribution of the equivalent stress, the equivalent plastic strain and the state variable were calculated for each of the dies. The forward differences (FDM) as well as the central differences methods (CDM) were used to calculate the sensitivities. Since the FDM and the CDM results agreed very well with each other, only the FDM results are reported below. From the sensitivities obtained, the gradient in the direction of a particular $\alpha_i$ is calculated by dividing the corresponding sensitivities by the increment $\Delta \alpha_i$.

The sensitivity calculations were also performed using the direct differentiation method (DDM) developed in this work. The variation in the die shape obtained by perturbing one of the die parameters in the initial die design provides the driving force for the sensitivity equilibrium equations. As in the finite differences method, each of the six parameters $\alpha_i$ were varied from their values at the reference die, one
at a time, by $1 \times 10^{-4}$. For each of the die parameters, the sensitivity field problem is solved one by one and the resulting sensitivity fields of various distributions at the exit are tabulated. Equation (4.37) is then used to calculate the required gradients. The main advantage of the analysis developed here is that the sensitivities of different fields can be calculated simultaneously for a given reference die.

The comparative results are shown below in Table (4.1) for the quantities $ds/d\alpha_1$ and $d\bar{\sigma}/d\alpha_1$. Table (4.2) presents the results for the quantities $ds/d\alpha_2$ and $d\bar{\sigma}/d\alpha_2$, while Table (4.3) presents the results for $ds/d\alpha_3$ and $d\bar{\sigma}/d\alpha_3$. The continuum based results agree very well with the finite differences solutions. Similar accuracy is observed for the gradients with respect to the other three parameters $\alpha_4$, $\alpha_5$ and $\alpha_6$ presented in Tables (4.4)-(4.6). Thus, the sensitivity analysis developed in this work achieves a high accuracy along with significant improvements in the required computing time over a finite difference method.

### 4.6.2 Solution to the optimum die design problems

Here, an optimization problem is posed for the selection of the die parameters, $\alpha_i$, $i=1,2,...$, such that certain criteria are satisfied for the material state in the final product. The discretized problems of interest have the following general form:

$$\min_{\alpha_i} \{\chi - \hat{\chi}\}^T \{\chi - \hat{\chi}\}$$

(4.45)\]

where $\{\hat{\chi}\}$ are the desired nodal values of the material state at the exit. An iterative procedure is followed to obtain the solution of this problem. Let $\{\alpha^r\}$ be the solution at the step $r$. The sensitivity of $\{\chi\}$ is determined as $[S_\alpha]\{\delta\alpha\}$. Then at step $r+1$
Table 4.1: Comparison of the gradients $ds/d\alpha_1$ and $d\sigma/d\alpha_1$ at the exit calculated using FDM and DDM

<table>
<thead>
<tr>
<th>$x$</th>
<th>$ds/d\alpha_1$ (DDM)</th>
<th>$ds/d\alpha_1$ (FDM)</th>
<th>$d\sigma/d\alpha_1$ (DDM)</th>
<th>$d\sigma/d\alpha_1$ (FDM)</th>
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Table 4.2: Comparison of the gradients \( ds/d\alpha_2 \) and \( d\sigma/d\alpha_2 \) at the exit, calculated using FDM and DDM

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<th>( ds/d\alpha_2 ) (FDM)</th>
<th>( d\sigma/d\alpha_2 ) (DDM)</th>
<th>( d\sigma/d\alpha_2 ) (FDM)</th>
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</tr>
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</tr>
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Table 4.3: Comparison of the gradients $ds/d\alpha_3$ and $d\sigma/d\alpha_3$ at the exit, calculated using FDM and DDM

<table>
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<tr>
<th>$x$</th>
<th>$ds/d\alpha_3$ (DDM)</th>
<th>$ds/d\alpha_3$ (FDM)</th>
<th>$d\sigma/d\alpha_3$ (DDM)</th>
<th>$d\sigma/d\alpha_3$ (FDM)</th>
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<tbody>
<tr>
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</tr>
<tr>
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Table 4.4: Comparison of the gradients $ds/d\alpha_4$ and $d\sigma/d\alpha_4$ at the exit, calculated using FDM and DDM

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<th>$ds/d\alpha_4$ (DDM)</th>
<th>$ds/d\alpha_4$ (FDM)</th>
<th>$d\sigma/d\alpha_4$ (DDM)</th>
<th>$d\sigma/d\alpha_4$ (FDM)</th>
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</tr>
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<td>1.6403E+00</td>
</tr>
<tr>
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<tr>
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Table 4.5: Comparison of the gradients $ds/d\alpha$ and $d\sigma/d\alpha$ at the exit, calculated using FDM and DDM.

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<th>$ds/d\alpha$ (DDM)</th>
<th>$ds/d\alpha$ (FDM)</th>
<th>$d\sigma/d\alpha$ (DDM)</th>
<th>$d\sigma/d\alpha$ (FDM)</th>
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Table 4.6: Comparison of the gradients $ds/d\alpha_6$ and $d\sigma/d\alpha_6$ at the exit, calculated using FDM and DDM

<table>
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<th>$x$</th>
<th>$ds/d\alpha_6$ (DDM)</th>
<th>$ds/d\alpha_6$ (FDM)</th>
<th>$d\sigma/d\alpha_6$ (DDM)</th>
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<td>5.8632E-02</td>
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</tr>
</tbody>
</table>
we have,

\[ \{ \chi^{r+1} \} = \{ \chi^r \} + [S_\alpha] \{ \delta \alpha \} \]  \hspace{1cm} (4.46)

Substituting the above equation into equation (4.45) and solving, results in the following step size:

\[ \{ \delta \alpha \} = - [s_\alpha^T s_\alpha]^{-1} \left( [s_\alpha^T] \{ \chi^r - \hat{\chi} \} \right) \]  \hspace{1cm} (4.47)

With the calculated step size, the die is updated and the iterative process is repeated until certain convergence criteria are satisfied. Here, it should be emphasized that the increment \{ \delta \alpha \} is the step size in the optimization scheme and is not related with the increment used in equation (4.37) for the calculation of the gradient from the sensitivities. In the following two subsections, two different problems are considered. In the first problem, \{ \chi \} represents the distribution of \( s \) at the exit, while in the second problem it represents the distribution of \( \sigma \) in the exit cross-section.

**Optimization of the die to achieve uniform distribution of \( s \) at the exit**

The sensitivity analysis presented earlier is used to design a die that results in a distribution of the internal state variable at the exit that is as uniform as possible. The objective function is here taken to be the square of the standard deviation of \( s \).

Let the cross sectional area where \( s \) is monitored be denoted as \( S \). The cross sectional area \( A \) is given as,

\[ A = \int_S dS \]  \hspace{1cm} (4.48)
Then, the optimization problem is defined as:

Obtain the die that results in the least standard deviation of the distribution \( s \) in the final product

\[
\min_{b(x, y)} \frac{1}{A} \int_{\mathcal{S}} (s(x, y; x_{\text{exit}}, t_f) - \bar{s})^2 \, dS
\]  

(4.49)

where \( \bar{s} \) is the (unknown) average distribution at the exit cross section, i.e.

\[
\bar{s} = \frac{1}{A} \int_{\mathcal{S}} s(x, y_{\text{exit}}; t_f) \, dS
\]

(4.50)

where \( y_{\text{exit}} = c + l + d \).

The above objective function is evaluated by taking discrete data points at the exit cross section. Then, the discretized problem can be written as

\[
\min_{\alpha_i} \sum_{i=1}^{n} W_i \left( s(x_i; y_{\text{exit}}, t_f) - \bar{s} \right)^2
\]

(4.51)

where \( W_i \) are the weights that result while performing the area integration at the exit. The method detailed in the previous section is employed to obtain the search direction and the step size. If the maximum of the \( \delta \alpha_i \) is above a critical value (here taken as 0.05), the parameters are scaled by the factor \( \text{(critical value}/\max(\alpha_i)) \) so that the solution does not cross the design space under consideration. After taking this step, the die parameters are updated and the sensitivity calculations are continued. The search is concluded when the successive values of the standard deviation of the distribution of \( s \) falls within 2\%. For the above problem, it took 7 iterations to achieve the optimal die. Figure (4.6.2) shows the initial and the optimal dies along with 2 intermediate dies. Figure (4.6.2) shows the distribution of \( s \) at the exit for both the initial and optimal dies, while Figure (4.6.2) presents
Figure 4.7: The initial and the intermediate dies, and the optimal die that results in near uniform distribution of $s$ in the final product.

The contour distribution of $s$ within the workpiece obtained with the optimum die and at the end of the process ($t_f = 350$ sec). It can be clearly seen from the last two Figures that a considerable improvement in the distribution of $s$ in the final product has been achieved when the optimal die is adopted.

**Optimization of the die to achieve uniform distribution of $\bar{s}$ at the exit**

Here the optimization problem can be defined as:

*Obtain the die that results in the least standard deviation of the distribution $\bar{s}$ in the final product*

\[
\min_{b(x, y)} \int_{\mathcal{S}} \frac{1}{A} \left( \bar{s}(x, y_{exit}, t_f) - \bar{s} \right)^2 dS
\]  

(4.52)
Figure 4.8: The distribution of $s$ at the exit for the initial die and the optimal die.

Figure 4.9: The contour distribution of $s$ in the workpiece at the final time using the optimum die.
Figure 4.10: The initial and the intermediate dies, and the optimal die that results in near uniform distribution of $\tilde{\sigma}$ in the final product.

where $\tilde{\sigma}$ is the average of the distribution of $\tilde{\sigma}$ at the exit cross section,

$$\tilde{\sigma} = \frac{1}{A} \int_{S} \tilde{\sigma} (x, y, x_i, t_f) \, dS \tag{4.53}$$

The same numerical procedure was adopted as in the previous problem. Convergence takes place in 7 iterations. Figure (4.6.2) shows the initial and the optimal dies along with 2 intermediate dies. Figure (4.6.2) shows the distribution of $\tilde{\sigma}$ at the exit for both the initial and optimum dies. Finally, Figure (4.6.2) presents the contour distribution of $\tilde{\sigma}$ within the workpiece obtained with the optimum die and at the end of the process.
Figure 4.11: The distribution of $\sigma$ at the exit for the initial die and the optimal die.

Figure 4.12: The contour distribution of $\sigma$ in the workpiece at the final time using the optimum die.
4.7 Conclusions

The sensitivity results obtained using the method developed in this paper agree very well with the solutions calculated using finite differences approximations. The large deformation solver developed for the sensitivity analysis is highly robust and capable of using large step sizes. The direct Newton-Raphson analysis converges in about 5 or 6 iterations per time step with an accuracy of 98% (based on the convergence of the total work done). For \( n \) finite element nodes in the given problem, in order to calculate the sensitivities of the displacements using the Finite Differences approach, we need to solve algebraic equations of size \( (2n \times 2n) \), 30 times per time step (assuming that there are six unknown die parameters). However, in the present analysis only 11 to 12 linear equation solution steps are required. Five to six of these steps are required in the Newton-Raphson scheme for the direct analysis. The remaining 6 steps are required in the sensitivity analysis for the solution of six algebraic systems of linear equations. Thus, the problem can be solved approximately 66.7% faster and a considerable amount of time and computer resources can be saved.

This example demonstrates that it is a powerful tool for handling complicated die design problems.
Chapter 5

Preform Design Problems

5.1 Introduction

During a open or a closed die forging operations, a workpiece of simple geometry is deformed to achieve the desired final shape. Often the only factors known to the process engineer are the final component shape and the material with which it has to be made. It becomes the job of the engineer to choose the initial preform shape, the shape of the die and other process parameters that will result in the required final shape. The starting workpiece shape for a typical forming operation is usually a simple shape such as a bar with a circular or rectangular cross-section. Such a simple shape cannot be deformed to achieve complex geometry through a single stage forming operation. Usually, more than one stage has to be involved in forming the product.

In this work, we do not deal with a multi-stage forming problem. Our aim here
is merely to demonstrate a simple preform design by proposing an optimization problem. For carrying out the optimization by using a sequential search method, we need to evaluate the gradients of the objective function. To compute the gradients of the objective function in the finite dimensional design space, we develop a sensitivity analysis scheme.

In this chapter, we provide a detailed derivation of the shape derivative of the equilibrium equations for the motion of the body. Combined with the relations derived for the shape derivatives of the constitutive variables in the previous chapter, we present the governing equations for the sensitivity fields. Then we present the boundary conditions for the shape sensitivity problem. Finally, we solve a simple preform design problem in open die forging of an axisymmetric cylinder. The objective here is to design a preform shape that will result in a product with no barrelling under forging by a flat die.

5.2 Shape derivative of the equilibrium equations

In order to solve for the shape sensitivity deformation field, we need to solve the shape derivative of the equilibrium equation with appropriate boundary conditions. In the following, a principle of virtual work like equation is developed for obtaining $\vec{F}$. The boundary conditions for this problem are treated in the next section. Unlike the direct problem handled in Chapter 1, we have formulated the sensitivity problem using a total Lagrangian framework.

The shape derivative of the equilibrium equations (equation 2.5) results in the
following equation:

\[
\nabla \cdot \mathbf{P} + f = 0 \quad \forall \ X \in \ B_0 \quad \text{and} \quad \forall \ t \in [0, t_f] \tag{5.1}
\]

In this work, we ignore the body forces. Thus, from equation (3.16), we express the above equation as,

\[
\nabla \cdot \left( \mathbf{P} - P L_R^T \right) + P \left[ \nabla \cdot L_R^T \right] = 0 \tag{5.2}
\]

Based on the definition of \( P \) from equation (2.6), \( \mathbf{P} \) is calculated as

\[
\mathbf{P} = \det F \left[ \text{tr} \left( \mathbf{F}^{-1} \right) T + \mathbf{F} T - T \left( \mathbf{F}^{-1} \mathbf{F} \right)^T \right] F^{-T} \tag{5.3}
\]

In the above equation, \( T \) is given by equation (3.17).

A variational form of the shape derivative of the equilibrium equation can now be posed as:

\[
\int_{B_0} \left( \nabla \cdot \left( \mathbf{P} - P L_R^T \right) + P \left[ \nabla \cdot L_R^T \right] \right) \cdot \mathbf{u} \ dV_0 = 0 \tag{5.4}
\]

where \( \mathbf{u} \) is a kinematically admissible field expressed over the reference configuration. Each term in the equation (5.4) is integrated by parts.

\[
\int_{B_0} \left( \nabla \cdot \mathbf{P} \right) \cdot \mathbf{u} \ dV_0 = \int_{\partial B_0} \mathbf{P} \cdot m \cdot \mathbf{u} \ dA_0 - \int_{B_0} \mathbf{P} \cdot \nabla \mathbf{u} \ dV_0
\]

where \( m \) is the unit outward normal at \( X \) in the reference configuration. \( n \) is the unit normal vector in the current configuration corresponding to \( m \). From the shape derivative of the traction vector \( \mathbf{t} \), we obtain:

\[
\mathbf{t} = (\mathbf{P} n) = T \mathbf{n} + \mathbf{t} n
\]
The shape derivative of the normal vector $\mathbf{n}$ can be obtained from transforming this vector field to the reference configuration $\mathbf{B}_R$. Based on similar calculations in Chapter 4, one can show that,

\[
\ddot{\mathbf{n}} = - \left( \mathbf{F}^{-1} \mathbf{F}^T \right)^T \mathbf{n} - \left( \mathbf{F}^{-T} \mathbf{L}_R^T \mathbf{F}^T \right) \mathbf{n} + \left\{ \mathbf{F}^{-1} \mathbf{F}^T \right\} \cdot (\mathbf{n} \otimes \mathbf{n}) \mathbf{n} + \left\{ \mathbf{F} \mathbf{L}_R \mathbf{F}^{-1} \right\} \cdot (\mathbf{n} \otimes \mathbf{n}) \mathbf{n}
\]

Therefore,

\[
\ddot{\mathbf{T}} \mathbf{n} + \text{tr} \left( \mathbf{F}^{-1} \mathbf{F}^T \right) \mathbf{T} - \mathbf{T} \left( \mathbf{F}^{-1} \mathbf{F}^T \right)^T \mathbf{n} = \left\{ \mathbf{F}^{-1} \mathbf{F}^T \right\} \cdot (\mathbf{n} \otimes \mathbf{n}) \mathbf{n} + \left\{ \mathbf{F} \mathbf{L}_R \mathbf{F}^{-1} \right\} \cdot (\mathbf{n} \otimes \mathbf{n}) \mathbf{n} + \mathbf{T} \left( \mathbf{F}^{-T} \mathbf{L}_R^T \mathbf{F}^T \right) \mathbf{n}
\]

Thus,

\[
\int_{\partial \mathbf{B}_0} \mathbf{P} \mathbf{m} \cdot \mathbf{u} \, dA_0 = \int_{\partial \mathbf{B}_{n+1}} \frac{1}{\det \mathbf{F}} \mathbf{P} \mathbf{F}^T \mathbf{n} \cdot \mathbf{u} \, dA_{n+1}
\]

\[
= \int_{\partial \mathbf{B}_{n+1}} \mathbf{t} \cdot \mathbf{u} \, dA_{n+1}
\]

\[
+ \int_{\partial \mathbf{B}_{n+1}} \left\{ \mathbf{F} \mathbf{L}_R \mathbf{F}^{-1} \right\} \cdot (\mathbf{n} \otimes \mathbf{n}) \mathbf{n} + \mathbf{T} \left( \mathbf{F}^{-T} \mathbf{L}_R^T \mathbf{F}^T \right) \mathbf{n} \cdot \mathbf{u} \, dA_{n+1}
\]
Similarly,
\[
\int_{B_0} \left( \nabla \cdot (PL_R^T m) \right) \cdot \du \, dV_0 = \int_{\partial B_0} PL_R^T m \cdot \du_0 \, dA_0 - \int_{B_0} \left( PL_R^T \right) \cdot \nabla \du \, dV_0
\]
where,
\[
\int_{\partial B_0} PL_R^T m \cdot \du_0 = \int_{\partial B_{n+1}} \frac{1}{\det F} \left( PL_R^T n \cdot \du_{n+1} \right)
\]
\[
= \int_{\partial B_{n+1}} T \left( F^{-T} L_R^T F^T \right) n \cdot \du_{n+1}
\]
Thus, one finally arrives at the following variational shape sensitivity problem:

Calculate \( \hat{\mathbf{x}}(X,t) \) such that

\[
\int_{B_0} \hat{P} \cdot \nabla \du \, dV_0 - \int_{\partial B_{n+1}} \left( \left( F^\circ F^{-1} \right) \cdot (I - n \otimes n) \right) t \cdot \du_{n+1} =
\]
\[
\int_{B_0} \left( P \left[ \nabla \cdot L_R^T \right] \right) \cdot \du \, dV_0 + \int_{B_0} \left( PL_R^T \right) \cdot \nabla \du \, dV_0
\]
\[
+ \int_{\partial B_{n+1}} \left\{ \hat{t} - \left( FL_R F^{-1} \cdot (n \otimes n) \right) t \right\} \cdot \du_{n+1}
\]
for every “admissible” test function \( \du \).

**NOTE:** A variational sensitivity problem can also be derived by taking the shape derivative of the weak form of the direct problem (equation (2.46)). However, we have not followed this approach for deriving the weak sensitivity problem since it involves further mathematical complexities. In particular, one must consider the shape derivatives of the test function \( \du \) and perform shape differentiation of integrals.
In the above equation, the quantity \( L_R \) and \( \dot{\dot{t}} \) are the inputs to the shape sensitivity problem. A value of \( L_R \equiv 0 \) results in the above equation reducing to that of the parameter sensitivity equilibrium equation. \( L_R \) is defined through the reference map as given in equation (3.12). In using the finite element method to solve the above kinematic problem, both terms in the left hand side of equation (5.5) contribute to the stiffness matrix, while the first term also contributes to the force vector. The right hand side contributes only to the force vector. The stiffness matrix for the shape sensitivity problem (the left hand side of equation (5.5)) is not the same as the linearized tangent stiffness used in the direct kinematic analysis.

### 5.3 Initial conditions for the sensitivity problem

The following set of initial conditions are used in the direct analysis:

\[
\begin{align*}
T(X, 0) &= 0 \\
\overset{o}{F^p}(X, 0) &= I \\
s(X, 0) &= s_0
\end{align*}
\tag{5.6}
\]

The corresponding initial conditions for the sensitivity problem have as follows:

\[
\begin{align*}
\overset{o}{T}(X, 0) &= 0 \\
\overset{o}{F^p}(X, 0) &= 0 \\
\overset{o}{s}(X, 0) &= 0
\end{align*}
\tag{5.7}
\]
5.4 Boundary conditions for the shape sensitivity problem

The boundary conditions for the shape sensitivity problem are derived from the corresponding boundary conditions of the direct analysis. Some typical cases are examined below.

5.4.1 Free boundary

The boundary tractions are zero on a free boundary. The corresponding boundary condition for the sensitivity problem can be written as follows:

\[ \circ \mathbf{t} = 0 \]  

(5.8)

5.4.2 Boundary with prescribed displacement

On this boundary, for the direct problem,

\[ \mathbf{x} = \mathbf{x} (\mathbf{X}, t) \]  

(5.9)

where \( \mathbf{x} \) is a known function. Therefore, for the shape sensitivity problem,

\[ \circ \mathbf{x} = 0 \]  

(5.10)

5.4.3 Boundary conditions for a contact surface

By solving the direct problem, the tractions and the displacement on the contact boundary are completely determined. We need to provide the change in the boundary tractions and displacements that will be induced because of a change in the
initial preform shape. We make the following simplifying assumptions to determine
the boundary conditions for the sensitivity problem:

1. A particle that is in contact with the die at a particular time $t$ for a particular
preform is also in contact with the die at the time $t$ for another closely preform.

2. A small change in the preform shape does not alter the regions where sticking
friction and sliding is prevalent.

Let $\mathcal{D}$ denote the die surface. $\mathbf{x} \in \mathcal{D}$ if $b(\mathbf{x}) = 0$ where $b$ is the equation of the
die surface. Also, from the direct problem we have,

$$\partial B_{n+1} = (\partial B_{\text{stick}})_{n+1} \cup (\partial B_{\text{slide}})_{n+1}$$

where

$$(\partial B_{\text{slide}})_{n+1} = \partial B_{n+1} - (\partial B_{\text{stick}})_{n+1}$$

and

$$\partial B_{\text{stick}} \subseteq \mathcal{D} \text{ and } \partial B_{\text{slide}} \subseteq \mathcal{D}$$

Based on the assumptions we have made above, the effect of the variation in
the preform shape is to induce a change in the normal and the tangential traction
distribution along with a compatible change in the boundary displacement field. On
the boundary where sticking friction prevails, there is no relative motion between
the die and the workpiece. Therefore, on $\partial B_{\text{stick}}$, the boundary condition is given
by:

$$\mathbf{x}_{n+1} = \mathbf{x}_n \quad \text{for } \mathbf{x}_{n+1} \in (\partial B_{\text{stick}})_{n+1}$$  \hspace{1cm} (5.11)
On the boundary regions where sliding contact prevails, the shape sensitivities of the tangential and normal tractions are related to each other by the coefficient of friction $\mu_f$. Further, from the assumption (i), we have,

$$
\begin{align*}
    b(x) &= 0 \\
    b(x + \hat{x}) &= 0
\end{align*}
$$

Thus, on $\partial B_{\text{slide}}$, the boundary conditions are given as,

$$
\begin{align*}
    \nabla b \cdot \hat{x} &= 0 \\
    t_T &= \mu_f t_N
\end{align*}
$$

for $x_{n+1} \in (\partial B_{\text{slide}})_{n+1}$ (5.13)

Thus, the entire shape sensitivity problem has been posed in a complete infinite dimensional setting along with all the necessary initial and the boundary conditions.

### 5.5 Finite Element Implementation of the Preform Design Problem

The preform design problem involves choosing a design space, an objective function to optimize, a set of practical constraints, an optimization algorithm to evaluate the search direction and the step length and a code for evaluating the value of the objective function and the gradients of the objective function whenever the optimization algorithm needs them. As mentioned earlier, we employ the sensitivity analysis developed in this work to determine the gradients of the objective function. In heat transfer and solidification problems ([22] - [24]), techniques have been developed
for extracting the gradient information in infinite dimensional spaces starting from
the sensitivity problem by defining an auxiliary problem for the solution of the ad-
joint operator (to the gradient). In this work, even though the sensitivity analysis
was developed for arbitrary variations of $L_R$, corresponding schemes to extract the
gradients in infinite dimensional spaces have not yet been developed. Therefore,
we will be using only finite dimensional design spaces where the calculation of the
gradients from the sensitivity problem are straightforward.

The spatial and the temporal discretization of the shape sensitivity problem are
identical to the ones used in the direct problem. Cross triangular finite elements
are employed with linear shape functions. Since both the direct problem and the
shape sensitivity problem use Euler backward integration, large time stepping are
employed. Typically, a open die forging operation involving 50% of height reduction
is solved typically using 6 to 10 time steps.

5.5.1 Design space

As mentioned earlier, the sensitivity problem is driven by the value of the distri-
bution $L_R$. $L_R$ signifies the change in the shape of two nearby configurations. For
the linearization employed in the sensitivity analysis to be valid, the value of $L_R$
should be considerably smaller compared to the reference gradient $F_R$ in the sense
of the traditional metric. For solving the preform design problem, we choose a fi-
nite number of scalar parameters that define the shape of the preform. These scalar
parameters map the reference configuration to the material configuration. The refer-
ence configuration is selected as a unit square and likewise four distinct boundaries
are determined in the material configuration (preform). Each of the sides of the
unit square are mapped to a corresponding boundary in the preform by employing
the shape parameters. These shape parameters form the design space. After the
boundaries are mapped correctly, the mapping from the reference configuration to
the material configuration $\mathbf{X}$ is done through a linear transfinite mapping of the
boundary mappings [43].

The reference mapping so defined above is, by no means, unique. This just
happens to be a convenient definition for the class of problems considered in this
work. One can, if the situation demands it, define a unit circle and a corresponding
boundary map, followed by a consistent reference map for another class of problems.
Similar problems can be solved in three dimensions by starting out with a unit cube
or a unit sphere etc.

5.5.2 Evaluation of the gradients from the shape sensitivity
solutions

Let us suppose that the shape sensitivity problem has been solved for each of the
$\Delta \beta_i$, $i = 1, 2, \ldots, n$ and the sensitivities of a field $\mathbf{Z}(\mathbf{X}, t; \beta_1, \beta_2, \ldots, \beta_n)$ has been
determined. Then, one can write:

$$\ddot{\mathbf{Z}}(\mathbf{X}, t; \beta_1, \beta_2, \ldots, \beta_n, \Delta \beta_1, \Delta \beta_2, \ldots \Delta \beta_n) = \sum_{i=1}^{n} \frac{\partial \mathbf{Z}}{\partial \beta_i} \Delta \beta_i \quad (5.14)$$

Therefore,

$$\frac{\partial \mathbf{Z}}{\partial \beta_i} = \frac{\ddot{\mathbf{Z}}(\mathbf{X}, t; \beta_1, \beta_2, \ldots, \beta_n, 0, 0, \ldots, \Delta \beta_i, \ldots, 0)}{\Delta \beta_i} \quad (5.15)$$
The design gradient, $\nabla Z$ of any Lagrangian field $Z$ is defined in the above parameter space as,

$$\nabla Z = \left( \frac{\partial Z}{\partial \beta_1}, \frac{\partial Z}{\partial \beta_2}, \ldots, \frac{\partial Z}{\partial \beta_n} \right)$$

and its calculation requires the solution of $n$ sensitivity problems. In the $k^{th}$ sensitivity problem, all the incremental parameters except $\Delta \beta_k$ are zero. Therefore, the implementation of the complete preform design problem involves the solution of the direct problem and the $n$ linear sensitivity incremental problems corresponding to each of the shape parameters $\beta_k$.

## 5.6 Optimization algorithm

The optimization problem is solved by using a quasi Newton method called BFGS method. The idea here is to determine a search direction in the design space starting from an initial point and employ a line search along this search direction to find the optimal step length along this direction. The method is called quasi-Newton because the inverse Hessian is not calculated exactly but approximated by a sequence. Supposing the objective function is represented as below:

$$\min_\beta f(\beta)$$

The gradients are evaluated as detailed in the previous section. The descent direction is computed using the following algorithm. At the starting point,

$$H^{(0)} = I$$

$$d^{(0)} = -\nabla f(\beta^{(0)})$$
\[ \beta^{(1)} = \beta^{(0)} + l \, d^{(0)} \]

where \( l \) is the step length determined from a golden section line search. Subsequently, for the \( (j+1) \)th iteration,

\[
H^{(j+1)} = H^{(j)} + \frac{(\beta^{(j+1)} - \beta^{(j)}) \otimes (\beta^{(j+1)} - \beta^{(j)})}{(\beta^{(j+1)} - \beta^{(j)}) \cdot (\nabla f^{(j+1)} - \nabla f^{(j)})} \left[ H^{(j)} \left[ \nabla f^{(j+1)} - \nabla f^{(j)} \right] \right] \otimes \left[ H^{(j)} \left[ \nabla f^{(j+1)} - \nabla f^{(j)} \right] \right] \\
- \frac{(\nabla f^{(j+1)} - \nabla f^{(j)}) \cdot \left( H^{(j)} \left[ \nabla f^{(j+1)} - \nabla f^{(j)} \right] \right)}{(\nabla f^{(j+1)} - \nabla f^{(j)}) \cdot \left( H^{(j)} \left[ \nabla f^{(j+1)} - \nabla f^{(j)} \right] \right)} \left[ H^{(j)} \left[ \nabla f^{(j+1)} - \nabla f^{(j)} \right] \right] + \left( (\nabla f^{(j+1)} - \nabla f^{(j)}) \cdot \left( H^{(j)} \left[ \nabla f^{(j+1)} - \nabla f^{(j)} \right] \right) \right) u \otimes u \quad (5.16)
\]

where,

\[
u = \frac{(\beta^{(j+1)} - \beta^{(j)})}{(\beta^{(j+1)} - \beta^{(j)}) \cdot (\nabla f^{(j+1)} - \nabla f^{(j)})} \left[ H^{(j)} \left[ \nabla f^{(j+1)} - \nabla f^{(j)} \right] \right] \\
- \frac{(\nabla f^{(j+1)} - \nabla f^{(j)}) \cdot \left( H^{(j)} \left[ \nabla f^{(j+1)} - \nabla f^{(j)} \right] \right)}{(\nabla f^{(j+1)} - \nabla f^{(j)}) \cdot \left( H^{(j)} \left[ \nabla f^{(j+1)} - \nabla f^{(j)} \right] \right)} \left[ H^{(j)} \left[ \nabla f^{(j+1)} - \nabla f^{(j)} \right] \right] \\
d^{(j+1)} = -H^{(j+1)} \left[ \nabla f^{(j+1)} \right] \\
\beta^{(j+2)} = \beta^{(j+1)} + l \, d^{(j+1)}
\]

where \( l \) is the golden section search step length.

### 5.7 A numerical example

In this section, an example problem in open die forging. In the open die forging of a cylindrical workpiece, a preform shape is designed that results in a free surface as flat as possible. The material properties for Aluminum are described in Table (2.1) in Chapter 2.
Figure 5.1: Open die forging of a cylindrical billet by 44.44%

In a typical open die forging process of cylindrical workpieces with a flat die, because of the friction between the die and the workpiece, the final product has a considerable barrelling. The objective here is to design the preform shape that, after a fixed amount of height reduction, results in a product with a perfectly flat free surface. Preliminary solution for a simple cylindrical workpiece made of Aluminum resulted in considerable barrelling when its height was reduced by 44.44%, as seen in the Figure (5.1). Using a high coefficient of friction 0.4, the friction between the die and the workpiece was modeled as sticking friction.
5.7.1 Shape parametrization

The process is simulated using Finite Element Method and modeling the workpiece as axially symmetric. Because of the symmetry of the problem, only one quadrant of the geometry is considered for analysis. The free surface shape in the initial preform is taken as the design parameter in the problem. A reference configuration of unit square is chosen and the free boundary is mapped on to a corresponding boundary in the reference configuration as shown in Figure (5.2).

The material configuration has four distinct boundaries of which one of them is the design surface and the rest are straight lines. This curved boundary is parametrized by $n$ shape parameters $\mathbf{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ and a family of $n$ basis functions $(\phi_1, \phi_2, \ldots, \phi_n)$. The curved boundary is parametrized using a scalar
parameter \( \alpha \in [0, 1] \) as

\[
X_1 = r(\alpha) = \sum_{i=1}^{n} \beta_i \phi_i(\alpha)
\]

\[
X_2 = c \, \alpha
\]

(5.17)

where \((X_1, X_2)\) is a point on this curved boundary and the function \(r\) satisfies the following conditions:

\[
r(0) = a
\]

\[
r(1) = b
\]

Now, employing a linear transfinite mapping, we obtain the reference map \(\hat{X}\) as

\[
X_1 = Y_1 r(Y_2)
\]

\[
X_2 = c \, Y_2
\]

(5.18)

Thus, we can express \(F_R\) as,

\[
F_R = \begin{bmatrix}
    r(Y_2) & Y_1 r'(Y_2) \\
    0 & c
\end{bmatrix}
\]

(5.19)

where \(r'\) denotes the derivative of the function \(r\). If one of the shape parameters, say, \(\beta_k\) is perturbed by \(\Delta \beta_k\) while keeping all the other parameters constant, \(\hat{F}_R\) can be written as,

\[
\hat{F}_R = \Delta \beta_k \begin{bmatrix}
    \phi_k(Y_2) & Y_1 \phi'_k(Y_2) \\
    0 & 0
\end{bmatrix}
\]

(5.20)
Thus, the relationship between $L_R$ and $\beta$ for this specific case can be written as,

$$
L_R = \frac{\Delta \beta_k}{c r(Y_2)} \begin{bmatrix}
  c \phi_k(Y_2) & Y_1 \{ r(Y_2)\phi_k'(Y_2) - r'Y_2\phi_k(Y_2) \}
  0 \\
  0 & 0
\end{bmatrix}
$$

(5.21)

### 5.7.2 Free surface parametrization

In the previous section, the free surface was parametrized by the equation (5.17).

For solving the optimization problem, we employ 3 pieces of cubic splines to patch this surface, with 7 shape parameters. Thus,

$$
r(\alpha) = \sum_{i=1}^{7} \beta_i \phi_i(\alpha)
$$

(5.22)

where each of the functions $\phi_i$ are of the form,

$$
\phi_i(\alpha) = a_0 + a_1 \alpha + a_2 \alpha^2 + a_3 \alpha^3
$$

(5.23)

The shape parameters we consider are the following.

$$
\begin{bmatrix}
  \beta_1 \\
  \beta_2 \\
  \beta_3 \\
  \beta_4 \\
  \beta_5 \\
  \beta_6 \\
  \beta_7
\end{bmatrix} = 
\begin{bmatrix}
  r(0) \\
  r(1/3) \\
  r'(1/3) \\
  r(2/3) \\
  r'(2/3) \\
  r(1) \\
  r'(1)
\end{bmatrix}
$$

(5.24)

The basis functions derived using the above data, along with the condition $r'(0) = 0$ are given in Appendix F.
5.8 Optimization to minimize barelling

Let the free surface of the deformed body in the final configuration be denoted by \( \Gamma_f \). Then, the optimization problem can be written as

\[
\min_{\beta_i} \int_{\Gamma_f} (\mathbf{a} \cdot \mathbf{e}_r - r_0)^2 \, dl
\]

Subject to the constraint

\[
\int_{B_0(\beta_i)} dV = V_0
\]

where \( \mathbf{e}_r \) is the unit radial vector, \( V_0 \) the volume of the cylinder and \( r_0 \) the radius of the cylinder that would have resulted while upsetting a cylindrical piece of volume \( V_0 \) under frictionless conditions. In order to obtain the optimal solution for this problem, we need to evaluate the gradients of \( \mathbf{a} \in \Gamma_f \) with respect to the parameters \( \beta_i \). We employ the shape sensitivity analysis developed in the previous sections to obtain the shape sensitivities and from them we compute the gradient of the objective function.

The finite element mesh is made fine in the region where fold over occurs. The optimization is performed using a modified BFGS algorithm described in the previous sections. The solution is assumed to converge when \( (\|\Delta \beta_i\|/\|\beta_i\| < 1\% \) The optimal solution was obtained in 8 iterations. The optimal solution is given by
Figure 5.3: Undeformed and deformed configurations of the initial preform shape for a 44.44% height reduction

\[
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7
\end{bmatrix} =
\begin{bmatrix}
5.748 \\
5.876 \\
0.601 \\
6.052 \\
1.204 \\
6.003 \\
-10.033
\end{bmatrix}
\]

The initial preform shape is shown in Figure (5.3) and the optimal preform in Figure (5.6). Two intermediate results are shown in Figures (5.4) and (5.5).
Figure 5.4: Undeformed and deformed configurations of the preform shape after 3 iterations

Figure 5.5: Undeformed and deformed configurations of the preform shape after 5 iterations
Figure 5.6: Undeformed and deformed configurations of the optimal preform shape

5.9 Conclusions

The simple example carried out above shows that the shape sensitivity analysis developed in this work is an able method to carry out preform design problems. In case of the above problem, with an even finer mesh near the fold over region and a better representation of the free surface, at the expense of higher computational costs, one could obtain in a fine detail the preform shape that leads to a perfectly flat free surface in the final product. However, even in that case, the material properties will not be uniformly distributed in the final product.
Chapter 6

Summary, Conclusions and Suggestions

In this chapter, we present a summary of the study conducted in this work, followed by the conclusions that can be drawn from this study. We finally end this chapter with suggestions for future work that can augment and improve on the current findings.

6.1 Summary of the direct deformation problem

The current study did not attempt to provide a comprehensive direct simulation code which could be used for solving all kinds of metal forming problems. Instead, it was an effort to put together a lowest common denominator that could solve a simplified forming problem. Several assumptions were made about the material under deformation. The solid was assumed isotropic, and the plastic deformation was
considered rate-dependent with a single scalar variable. The deformations were assumed isothermal. An Euler-backward time-integration scheme with a radial return method was employed to integrate the evolution equations.

An updated Lagrangian reference frame setup was employed for the analysis of the motion of the body. The quasi-static boundary value problem, developed from the equilibrium equations were solved using a finite element scheme. The nonlinear boundary value problem was solved through a Newton-Raphson iterative scheme. A consistent tangent modulus was derived for achieving better convergence.

All of the above theory were already available in the literature. This work merely attempted to develop a finite element code based on the existing literature, which could then be used as a tool for solving die and preform design problems.

Initially, the code developed could handle only explicit boundary conditions such as known boundary displacements and traction conditions. This severely restricted the use of the code as most of the practical forming problems involved contact between a die and the workpiece. Contact conditions are implicit in nature and modeling contact in forming is a rather complex field. After the development of the direct simulation code, die and preform design problems were attempted within the constraints of the code developed. When more complicated examples were proposed for the design problem, the direct simulation code had to be modified to incorporate the implicit contact conditions.

The contact between the die and the workpiece was modeled based on the some existing research in the literature. In the method employed, augmented Lagrangian functions are employed to enforce the contact and friction constraints. A gap func-
tion is defined which provides a measure of the impenetrability constraint. Coulomb friction law is assumed to exist in the die-workpiece interface. The friction modeling introduces another level of iteration to the direct simulation code.

The modified direct simulation code performed adequately well in frictionless cases. However, when friction was introduced at the die-workpiece interface, the convergence achieved was very poor. In addition, the force profile as a function of the displacement was not smooth and contained oscillations. The applicability of the code developed to solve complex contact and frictional problems is not very sound.

6.2 Summary of the sensitivity problems

Sensitivity operators were rigorously defined as directional derivatives as opposed to the practice of using partial derivatives. Distinctions were made between the shape sensitivities and parameter sensitivities.

Direct differentiation of the strong form of the governing equations for the direct problem were employed to obtain governing equations for the sensitivity problem. An Euler backward integration scheme, consistent with the direct problem, was proposed for integrating the sensitivity evolution equations. Since the sensitivity fields are related to each other linearly, the overall sensitivity problem is also linear in nature and can be solved without resorting to any iterative schemes.

In some literature, a method is proposed whereby, for the sensitivity problem, the "stiffness matrix" is taken to be the tangent stiffness in the last Newton-Raphson
iteration of the corresponding direct problem. It is our conclusion that such a scheme is not correct. Such a scheme, while purporting to be computationally efficient, does not seem to explain clearly, how the corresponding “force vector” is computed.

We also demonstrated the reason as to why shape sensitivities had to be handled separate from the parameter sensitivities, as the sensitivity of the equilibrium equations, and consequently the weak form, differed from each other considerably. Also, by means of the example problems worked out, the boundary conditions were evaluated for the parameter sensitivity problem. In case of the shape sensitivity problem, it was clearly demonstrated as to how to obtain the driving force for such a problem.

The sensitivity constitutive equations developed are dependent on a particular constitutive model. However, the sensitivities of the kinematic and the equilibrium equations are completely general in spirit. If another constitutive model is attempted for a new material, one needs only to obtain the corresponding equations for the sensitivity problem.

6.3 Summary of the preform and die design examples

Admittedly, the example problems worked out for the preform and die design do not give rise to any revolutionary ideas. The problems are rather trivial in nature. The lack of more powerful examples for the die and preform design problems have more to do with the absense of a good direct simulation code capable of solving the complex
direct forming problem. Commercial codes are can be used for this purpose provided the commercial codes are flexible enough to allow the user to provide a material model, as well as allow the user to program the sensitivity problem together with the direct modeling. This is because both the direct and the sensitivity evolution equations have to be integrated over the discrete time intervals, and solution to the sensitivity problem requires the solution to the direct problem. Storing the solution to the direct problem at all time steps will be too cumbersome and space intensive. Therefore, this rules out almost all the commercial codes available.

Even though the sensitivity problems were developed so that the perturbation on the die or preform shape can be an infinite dimensional function, optimization was performed on finite dimensional spaces. This was done so that gradient information could be extracted from the sensitivity information.

The examples worked out, though simple, do indeed point out the power of the method in obtaining an optimal solution that improves upon the initial guess substantially.

6.4 Suggestions for future work

The wish list for future work could be almost endless. The direct deformation code can be overhauled and improved considerably to take into effect contact and friction in a comprehensive manner. Anisotrophy in material properties could be incorporated. Thermo-mechanical analysis can be incorporated to better model the real process, which usually sees a substantial increase in the temperatures which in
turn affect the stresses and the state variables.

More work needs to be done in studying the boundary conditions for the sensitivity problem especially under implicit contact. In particular, similar to the contact conditions for the direct simulation problems, it might be possible to come up with equations that better describe the sensitivities of the boundary displacements and tractions near the contact regions.

An adjoint problem could be studied to develop strategies for computing the gradient of various fields in an infinite dimensional settings. This could considerably improve the solutions for several design problems.

The topic die and preform design is very important as a comprehensive study of this will result in considerable savings in cost and time to the metal forming industry. Therefore it is strongly recommended that a systematic study of this topic be undertaken in this area.
Appendix A

Radial Return Mapping

From the definition of the trial elastic deformation gradient, we have,

$$ F^e = F^e_{n+1} \Delta \dot{F}^p $$

The uniqueness of the polar decomposition implies that

$$ R^e = R^e_{n+1} $$

and

$$ U^e = U^e_{n+1} \Delta \dot{F}^p $$

Taking logarithms of the above equation, we obtain

$$ \dot{E}^e_{n+1} + \Delta t \, \dot{D}^p_{n+1} (\ddot{T}^e_{n+1} \cdot s_{n+1}) = \ddot{E}^e_{s} $$

From the hyperelastic model, the definition of the trial stress and the flow rule, we obtain:

$$ \ddot{T}_{n+1} = \mathcal{L}^e [\ddot{E}^e_{n+1}] $$
\[
= \mathcal{C} \left[ \hat{E}^n \right] - \mathcal{C} \left[ \Delta t \, \tilde{D}_n \tilde{T}'_{n+1} \left( \tilde{T}'_{n+1}, s_{n+1} \right) \right]
\]
\[
= \tilde{T}_s - 2\mu \sqrt{\frac{3}{2}} \frac{\tilde{e}^p}{\tilde{\sigma}} \Delta t \tilde{N}_n \left( \tilde{T}'_{n+1}, s_{n+1} \right)
\]

From the definition of \( \tilde{N}_n \), we have,
\[
\tilde{T}'_{n+1} = \tilde{T}_s - 3\mu \frac{\tilde{e}^p}{\tilde{\sigma}} \frac{\tilde{T}'_{n+1}}{\tilde{\sigma}_{n+1}} \quad (A.1)
\]

Taking the trace and the deviatoric part of the above equation, we obtain:
\[
\tilde{p}_{n+1} = \tilde{p}_s \quad (A.2)
\]

and
\[
\left( 1 + \frac{3\mu \tilde{e}^p}{\tilde{\sigma}_{n+1}} \Delta t \right) \tilde{T}'_{n+1} = \tilde{T}_s' \quad (A.3)
\]

From the above, we see that \( \tilde{T}'_{n+1} \) and \( \tilde{T}_s' \) have the same direction. Taking the magnitude on both sides of the above equation results in
\[
\tilde{\sigma}_{n+1} \left( 1 + \frac{3\mu \tilde{e}^p}{\tilde{\sigma}_{n+1}} \Delta t \right) = \tilde{\sigma}_s \quad (A.4)
\]

which, upon simplification results in,
\[
\tilde{\sigma}_{n+1} - \tilde{\sigma}_s + 3\mu \Delta t f (\tilde{\sigma}_{n+1}, s_{n+1}) = 0 \quad (A.5)
\]

where
\[
\tilde{\sigma}_s = \sqrt{\frac{3}{2} \tilde{T}'_s}, \tilde{T}'_s
\]
\[
\tilde{\sigma}_{n+1} = \sqrt{\frac{3}{2} \tilde{T}'_{n+1}, \tilde{T}'_{n+1}}
\]

This, together with equation (2.22), forms a system of two nonlinear algebraic equations for \( \tilde{\sigma}_{n+1} \) and \( s_{n+1} \). The solution to these equations provides \( \tilde{T}'_{n+1} \). This
method is called the radial return method. The radial return factor is given by

$$\eta_{n+1} = \frac{\tilde{\sigma}_{n+1}}{\tilde{\sigma}_s}$$

Thus,

$$\tilde{T}'_{n+1} = \eta_{n+1} \tilde{T}'_s$$

and

$$\tilde{p}_{n+1} = -\frac{1}{3} \text{tr} (\tilde{T}_s)$$

Therefore,

$$\tilde{T}_{n+1} = \eta_{n+1} \tilde{T}'_s - \tilde{p}_s I$$

The Cauchy stress is then given by,

$$T_{n+1} = \left[\det \left(U^n_{n+1}\right)\right]^{-1} R^e_s \tilde{T}_{n+1} R^{eT}_s$$

Using the identities

$$\left[\det \left(U^n_{n+1}\right)\right]^{-1} = \left[\det \left(\exp \left(E^n_{n+1}\right)\right)\right]^{-1} = \exp \left(-\text{tr} \mathbf{E}^{e}_{n+1}\right)$$

and

$$\text{tr} \tilde{T}_{n+1} = 3\kappa \text{tr} \tilde{E}^{e}_{n+1} = -3\tilde{p}_{n+1}$$

and,

$$\left[\det \left(U^n_{n+1}\right)\right]^{-1} = \exp \left(\frac{\tilde{p}_{n+1}}{\kappa}\right) = \exp \left(\frac{\tilde{p}_s}{\kappa}\right)$$

we obtain,

$$T_{n+1} = \exp \left(\frac{\tilde{p}_s}{\kappa}\right) R^e_s \tilde{T}_{n+1} R^{eT}_s \quad (A.6)$$
Having calculated the Cauchy stress, we can update the $F^e_{n+1}$. From the radial return mapping, we have,

$$2\mu \left( \ln U^e_{n+1} - \frac{1}{3} \text{tr} \ln U^e_{n+1} I \right) = \eta_{n+1} \left( \ln U^e_{n} - \frac{1}{3} \text{tr} \ln U^e_{n} I \right)$$

This implies that $U^e_{n}$ and $U^e_{n+1}$ are in the same direction. From the spectral theorem, we obtain

$$U^e_{n+1} = \sum \lambda^i \mathbf{e}^i \otimes \mathbf{e}^i$$

and

$$U^e_{n} = \sum \lambda^i_{\ast} \mathbf{e}^i \otimes \mathbf{e}^i$$

After some simplifications, we have

$$\ln \left[ \frac{\lambda^i}{(\lambda^1 \lambda^2 \lambda^3)^{1/3}} \right] = \eta_{n+1} \ln \left[ \frac{\lambda^i_{\ast}}{(\lambda^1_{\ast} \lambda^2_{\ast} \lambda^3_{\ast})^{1/3}} \right]$$

or,

$$\frac{\lambda^i}{(\lambda^1 \lambda^2 \lambda^3)^{1/3}} = \left[ \frac{\lambda^i_{\ast}}{(\lambda^1_{\ast} \lambda^2_{\ast} \lambda^3_{\ast})^{1/3}} \right]^{\eta_{n+1}}$$

By incompressibility of $\Delta \tilde{F}^p$, we have

$$\lambda^1 \lambda^2 \lambda^3 = \lambda^1_{\ast} \lambda^2_{\ast} \lambda^3_{\ast}$$

Therefore,

$$\lambda^i = \frac{(\lambda^i_{\ast})^{\eta_{n+1}}}{(\lambda^1_{\ast} \lambda^2_{\ast} \lambda^3_{\ast})^{(\eta_{n+1} - 1)/3}} \quad (A.7)$$

from which $F^e_{n+1}$ can be calculated.
Appendix B

A Modified Newton-Raphson Scheme

In order to solve the nonlinear simultaneous system of algebraic equations arising out of the radial return method, the following method (given in [28]) is employed.

The equations to be solved are:

\[ s_{n+1} - s_n - \Delta t \cdot g(\tilde{\sigma}_{n+1}, s_{n+1}) = 0 \]  \hspace{1cm} (B.1)

\[ \tilde{\sigma}_{n+1} - \sigma_s + 3\mu \Delta t \cdot f(\tilde{\sigma}_{n+1}, s_{n+1}) = 0 \]  \hspace{1cm} (B.2)

These equations have to be solved by iterative means. To start with, an initial estimate of the solution is needed. From there on, the Newton-Raphson method is employed to solve the equations.

1. First, the tolerances required for the convergence on \( \tilde{\sigma}_{n+1} \) and \( s_{n+1} \) are fixed.

A good choice is

\[ \text{TOL}_s = 1 \times 10^{-5} \cdot s_n \]
2. Make an initial estimate of \((\sigma_{n+1}, s_{n+1})\) by solving the equations obtained by a forward gradient approximations of the governing equations. The functions \(f\) and \(g\) are approximated by Taylor series expansion about \((\sigma_{n}, s_{n})\). This results in

\[
\sigma_{n+1} \approx \sigma_{n} + \left\{ \frac{b_2 (\sigma_{n} - \sigma_{n} - 3 \mu \Delta t f (\sigma_{n}, s_{n})) - a_2 \Delta t g (\sigma_{n}, s_{n})}{b_2 a_1 + a_2 b_1} \right\}
\]

and

\[
s_{n+1} \approx s_{n} + \left\{ \frac{b_1 (\sigma_{n} - \sigma_{n} - 3 \mu \Delta t f (\sigma_{n}, s_{n})) + a_1 \Delta t g (\sigma_{n}, s_{n})}{b_2 a_1 + a_2 b_1} \right\}
\]

where

\[
a_1 \equiv 1 + 3 \mu \Delta t \frac{\partial f}{\partial \sigma_{n}}
\]

\[
a_2 \equiv 3 \mu \Delta t \frac{\partial f}{\partial s_{n}}
\]

\[
b_1 \equiv \Delta t \frac{\partial g}{\partial \sigma_{n}}
\]

\[
b_2 \equiv 1 - \Delta t \frac{\partial g}{\partial s_{n}}
\]

However, with certain models, (specifically, the sine hyperbolic model for Aluminium at 673 K) the value of \(\sigma_{n+1}\) calculated by the above equation becomes so high that further substitution in the functions \(f\) and \(g\) results in large numbers that cannot be handled by the computer. Hence in such cases the value of \(\sigma_{n+1}\) is taken to be equal to 0 or \(\sigma_{n}\).
3. Now level 1 iterations are performed to obtain subsequent values for $s_{n+1}$.

The iterations are performed till convergence is obtained on $s_{n+1}$. Consider a 

generic $k^{th}$ step. Assume that the function $f$ is invertible to give flow stress, 

ie., assume that there exists a function $f^{-1}$ such that 

$$\dot{\varepsilon}^p = f(\dot{\sigma}, s)$$

implies 

$$\dot{\sigma} = f^{-1}(\dot{\varepsilon}^p, s)$$

(a) Calculate an upper bound on $\dot{\sigma}_{n+1}$, by evaluating the smaller of the two 

quantities $\dot{\sigma}$ and $f^{-1}\left(\frac{\dot{\sigma}}{\Delta t}, s\right)$. Assign this value to $\sigma_n$.

(b) Fix the lower bound for $\dot{\sigma}_{n+1}$ as 0.

(c) Do level 2 iterations to update $\dot{\sigma}_{n+1}$ and to evaluate $(d\sigma_{n+1}/ds_{n+1})$ (de- 

dscribed below).

(d) Evaluate the error associated with $s_{n+1}$

$$E_s^k = s_{n+1}^k - s_n - \Delta t g_{n+1}^k$$

where

$$g_{n+1}^k = g(\dot{\sigma}_{n+1}^k, s_{n+1}^k)$$

(e) Compare this error with the tolerance $TOL_s$. Iteration is assumed to be 

converged if 

$$|E_s^k| \leq TOL_s$$

Then $s_{n+1} = s_{n+1}^k$ and $\dot{\sigma}_{n+1} = \dot{\sigma}_{n+1}^k$. Otherwise the iteration is continued.
(f) If the iteration continues, the correction $\Delta s^k$ is calculated.

$$\Delta s^k = \frac{-E^k_s}{dE^k_s/ds^k_{n+1}}$$

where

$$\frac{dE^k_s}{ds^k_{n+1}} = 1 - \Delta t \left\{ \frac{\partial g^k_{n+1}}{\partial s^k_{n+1}} + \frac{\partial g^k_{n+1}}{\partial \tilde{\sigma}^k_{n+1}} \frac{d\tilde{\sigma}^k_{n+1}}{d s^k_{n+1}} \right\}$$

where $d\tilde{\sigma}^k_{n+1}/ds^k_{n+1}$ is calculated during the level 2 iterations.

(g) The estimate for the next level iteration is

$$s^{k+1}_{n+1} = s^k_{n+1} + \Delta s^k$$

(h) The initial estimate of $\tilde{\sigma}^k_{n+1}$ to start the next level 2 iterations is

$$\tilde{\sigma}^{k+1}_{n+1} = \tilde{\sigma}^k_{n+1} + \frac{d\tilde{\sigma}^k_{n+1}}{ds^k_{n+1}} \Delta s^k$$

4. Level 2 iterations are performed to obtain the updates on $\tilde{\sigma}^k_{n+1}$. Consider the generic step $i$ within the $k^{th}$ loop above.

(a) Evaluate the error associated with $\tilde{\sigma}^k_{n+1}$

$$E_{\tilde{\sigma}}^{k,i} = \tilde{\sigma}^k_{n+1} - \sigma^k + 3\mu \Delta t f_{n+1}^{k,i}$$

where

$$f_{n+1}^{k,i} = g \left( \tilde{\sigma}^{k,i}_{n+1}, s^k_{n+1} \right)$$

(b) Compare this error with the tolerance. If

$$|E_{\tilde{\sigma}}^{k,i}| \leq \text{TOI}_{\sigma}$$
then the scheme has converged. Assign $\bar{\sigma}_{n+1}^{k} = \bar{\sigma}_{n+1}^{k,i}$. Evaluate

$$\frac{d\bar{\sigma}_{n+1}^{k}}{ds_{n+1}^{k}} = \frac{-3 \mu \Delta t \left( \partial f_{N+1}^{k,i} / \partial s_{n+1}^{k} \right)}{1 + 3 \mu \Delta t \left( \partial f_{N+1}^{k,i} / \partial \bar{\sigma}_{n+1}^{k} \right)}$$

for using in level 1 iterations.

(c) If not converged, Calculate the Newton-Raphson correction factor $\Delta \bar{\sigma}_{NR}^{k,i}$

$$\Delta \bar{\sigma}_{NR}^{k,i} = \frac{-E_{n}^{k,i}}{dE_{\bar{\sigma}}^{k,i} / d\bar{\sigma}_{n+1}^{k,i}}$$

where

$$dE_{\bar{\sigma}}^{k,i} / d\bar{\sigma}_{n+1}^{k,i} = 1 + 3 \mu \Delta t \frac{\partial f_{n+1}^{k,i}}{\partial \bar{\sigma}_{n+1}^{k,i}}$$

(d) Depending upon the sign of $\Delta \bar{\sigma}_{NR}^{k,i}$ update either the upper bound $\bar{\sigma}_u$ or the lower bound $\bar{\sigma}_l$ for $\bar{\sigma}_{n+1}$ and then calculate the maximum allowable correction, as given by a quasi-bisection scheme. If $\Delta \bar{\sigma}_{NR}^{k,i} < 0$, then

$$\bar{\sigma}_u = \bar{\sigma}_{n+1}^{k,i}$$

and

$$\Delta \bar{\sigma}_{max}^{k,i} = \frac{1}{2} \left( \bar{\sigma}_l - \bar{\sigma}_{n+1}^{k,i} \right)$$

If $\Delta \bar{\sigma}_{NR}^{k,i} > 0$, then

$$\bar{\sigma}_l = \bar{\sigma}_{n+1}^{k,i}$$

and

$$\Delta \bar{\sigma}_{max}^{k,i} = \bar{\sigma}_u - \bar{\sigma}_{n+1}^{k,i}$$

(e) Determine the correction to be used. If $|\Delta \bar{\sigma}_{NR}^{k,i}| > |\Delta \bar{\sigma}_{max}^{k,i}|$, then

$$\Delta \bar{\sigma}^{k,i} = \Delta \bar{\sigma}_{max}^{k,i}$$
else
\[ \Delta \tilde{\sigma}^{k,i} = \Delta \tilde{\sigma}_{NR}^{k,i} \]

(f) The estimate for the next iteration is
\[ \tilde{\sigma}_{n+1}^{k,i+1} = \tilde{\sigma}_{n+1}^{k,i} + \Delta \tilde{\sigma}^{k,i} \]

This algorithm works very well for the sine hyperbolic constitutive model.

For a rate independent model, one of the equations will become
\[ \tilde{\sigma}_{n+1} = \sigma_{n+1} \]

Hence, we have only one nonlinear equation to solve. In that case,
\[ F_{\sigma}^{i} = \tilde{\sigma}_{n+1}^{i} - \sigma_{s}^{i} + 3\mu \Delta t f_{n+1}^{i} \]

and
\[ \Delta \tilde{\sigma}_{n+1} = \frac{-F_{\sigma}^{i}}{1 + 3\mu \Delta t (\frac{\partial f}{\partial \sigma} + \frac{3\mu}{\sigma})} \]

This is continued till the convergence is achieved.
Appendix C

Linearization of the Internal Work Term in the PVW Equation

Differentiating the Piola-Kirchhoff stress, we obtain

\[ dP_u = d(\det F_u) \, T_{n+1} \, F_u^{-1} + \det F_u \, dT_{n+1} \, F_u^{-1} + \det F_u \, T_{n+1} \, dF_u^{-1} \]

Now,

\[ d(\det F_u) = \det F_u \, \text{tr} \left( dF_u \, F_u^{-1} \right) \]

and

\[ dF_u^{-1} = - \left( dF_u \, F_u^{-1} \right)^T \, F_u^{-1} \]

Hence

\[ dP_u = \det F_u \left[ \text{tr} \left( dF_u \, F_u^{-1} \right) \, T_{n+1} + dT_{n+1} - T_{n+1} \left( dF_u \, F_u^{-1} \right)^T \right] \, F_u^{-1} \]

To evaluate \( dT_{n+1} \), from the constitutive problem, we have

\[ dT_{n+1} = \exp \left( \frac{p_u}{\kappa} \right) \left( -\frac{1}{3\kappa} \, \text{tr}(d\bar{T}) \, R_s \, \bar{T}_{n+1} \right) + R_s \, \bar{T}_{n+1} \, d(R_s)^T + \]
\[ R_s^e d\mathbf{T}_{n+1}^T (R_s^e)^T + dR_s^e \mathbf{T}_{n+1}^T (R_s^e)^T \]  \hspace{1cm} (C.1)

Here

\[ d\mathbf{T}_{n+1} = \frac{\partial \mathbf{T}_{n+1}}{\partial \mathbf{E}_s^e} [d\mathbf{E}_s^e] = \mathbf{C} [d\mathbf{E}_s^e] \]

where the elastoplastic linearized moduli \( \mathbf{C} \) are defined as

\[ \mathbf{C} = \frac{\partial \mathbf{T}_{n+1}}{\partial \mathbf{E}_s^e} \]

Now we have to evaluate \( d\mathbf{E}_s^e \), \( d\mathbf{R}_s^e \) and the moduli \( \mathbf{C} \). Let us first consider \( d\mathbf{E}_s^e \).

\[ \mathbf{E}_s^e = \ln \mathbf{U}_s^e \]

By using first Padé approximation of \( \ln(\mathbf{U}_s^e) \), we can approximate \( \mathbf{E}_s^e \) as

\[ \mathbf{E}_s^e \approx 2(\mathbf{U}_s^e - \mathbf{I})(\mathbf{U}_s^e + \mathbf{I})^{-1} \]

This approximation is valid in the range of \( 0.7 < \lambda_s < 1.2 \), where \( \lambda_s \) is the stretch (eigen value) corresponding to \( \mathbf{U}_s^e \). Differentiating this, we obtain

\[ d\mathbf{E}_s^e = 4(\mathbf{U}_s^e + \mathbf{I})^{-1} \ d\mathbf{U}_s^e (\mathbf{U}_s^e + \mathbf{I})^{-1} \]

In order to evaluate \( d\mathbf{U}_s^e \), we have

\[ (\mathbf{U}_s^e)^2 = (\mathbf{F}_n^e)^T (\mathbf{F}_u^e)^T \mathbf{F}_u^e \mathbf{F}_n^e \]

which upon differentiation and simplification results in,

\[ \mathbf{U}_s^e d\mathbf{U}_s^e + d\mathbf{U}_s^e \mathbf{U}_s^e = 2 (\mathbf{F}_n^e)^T \text{sym} ((\mathbf{F}_u^e)^T d\mathbf{F}_u^e) \mathbf{F}_n^e \]

Here, “sym” refers to the symmetric part of the tensor. Consider the Biot strain,

\[ \mathbf{E}_{sB} \approx \mathbf{U}_s^e - \mathbf{I} \]

(C.4)
Upon expanding \((U^e_a)^{-1}\) in terms of \(E^B_a\) we arrive at

\[
(U^e_a)^{-1} = I - E^B_a + O\left(E^B_a^2\right)
\]

Now, premultiplying equation (C.3) by \((U^e_a)^{-1}\), substituting equation (C.4) and taking the symmetric part on both the sides, we arrive at

\[
dU^e_a = \text{sym}\left((U^e_a)^{-1} \left(F^e_n\right)^T \text{sym}\left(F^T_a dF^e_a\right) F^e_n\right) + O\left(E^B_a^2\right) \tag{C.5}
\]

Similarly for evaluating \(dR^e_a\)

\[
R^e_a U^e_a = F^e_a = F_a F^e_n
\]

Hence differentiating,

\[
dR^e_a U^e_a + R^e_a dU^e_a = dF_a F^e_n
\]

Now substituting equation (C.5) and simplifying the above, we arrive at:

\[
dR^e_a \left( R^e_a \right)^T = dF_a F^{-1}_a - R^e_a dU^e_a \left(U^e_a\right)^{-1} \left(R^e_a\right)^T \tag{C.6}
\]

Thus, \(dP_u\) can now be written as

\[
dP_u = \det F_a \left\{ \text{tr} \left(dF_a F^{-1}_a\right) T_{n+1} - T_{n+1} \left(dF_a F^{-1}_a\right)^T \right. \\
\left. -\text{tr} \left(\frac{1}{3\kappa} C \left[d\bar{E}^e_a\right]\right) T_{n+1} + \exp \left(\frac{p_a}{\kappa}\right) R^e_a C \left[d\bar{E}^e_a\right] \left( R^e_a \right)^T \right. \\
\left. + \left(dR^e_a \left( R^e_a \right)^T\right) T_{n+1} - T_{n+1} \left(dR^e_a \left( R^e_a \right)^T\right)^T \right\} F^{-T}_a \tag{C.7}
\]

We now have to calculate the linearized modulii \(C\). We will assume that the material is isotropic. Differentiating the equation (A.1) with respect to \(\bar{E}^e_a\), we obtain,

\[
C = \frac{\partial \bar{T}_a}{\partial \bar{E}^e_a} - 3\mu \Delta t \frac{\partial}{\partial \bar{E}^e_a} \left(\frac{f_a}{\sigma_a} \bar{T}_a\right)
\]
which results in

\[ \mathbf{C} = 2\mu \mathbf{I} + \left( \kappa - \frac{2}{3} \mu \right) \mathbf{I} \otimes \mathbf{I} - 3\mu \Delta t \left[ \frac{f}{\sigma_*} \frac{\partial \mathbf{F}'}{\partial \mathbf{E}_s} + \frac{\partial}{\partial \mathbf{E}_s} \left( \frac{f}{\sigma_*} \right) \otimes \mathbf{F}' \right] \]

\[ \text{(C.8)} \]

Here,

\[ \frac{\partial \mathbf{F}'}{\partial \mathbf{E}_s} = 2\mu \mathbf{I} + \frac{2}{3} \mu \mathbf{I} \otimes \mathbf{I} \]

\[ \text{(C.9)} \]

Differentiating the equations (2.22) and (2.28) and simplifying, we obtain

\[ d\sigma_{n+1} = c \, d\mathbf{\sigma}_s \]

\[ \text{(C.10)} \]

where

\[ c = \frac{b_2}{a_1 b_2 + a_2 b_1} \]

\[ a_1 \equiv 1 + 3\mu \Delta t \frac{\partial f}{\partial \sigma_{n+1}} \]

\[ a_2 \equiv 3\mu \Delta t \frac{\partial f}{\partial s_{n+1}} \]

\[ b_1 \equiv \Delta t \frac{\partial g}{\partial \sigma_{n+1}} \]

\[ b_2 \equiv 1 - \Delta t \frac{\partial g}{\partial s_{n+1}} \]

Differentiation of equation (2.25), results in

\[ \frac{\partial \sigma_s}{\partial \mathbf{E}_s} = \frac{3\mu \mathbf{F}'}{\sigma_*} \]

\[ \text{(C.11)} \]

and,

\[ \frac{\partial}{\partial \mathbf{E}_s} \left( \frac{f}{\sigma_*} \right) = 2\alpha \mu \mathbf{F}' \]

\[ \text{(C.12)} \]

where

\[ \alpha = \frac{1 - c}{2\mu \Delta t \sigma_*^2} - \frac{3f}{2\sigma_*^3} \]
Now we can write $\mathbf{C}$ as

$$
\mathbf{C} = 2\mu' \mathbf{I} + \left( \kappa - \frac{2}{3}\mu' \right) \mathbf{I} \otimes \mathbf{I} + \beta \mathbf{T}' \otimes \mathbf{T}'
$$

where

$$
\mu' = \eta_{n+1} \mu
$$

and

$$
\beta = \frac{3\mu}{\sigma^2} (c - \eta_{n+1})
$$
Appendix D

Some Commonly Occuring Linear Transformations

Consider the operation of pre-multiplying a second order tensor $X$ by a constant second order tensor $A$. This operation can also be represented as the dot product of a fourth order tensor operator $\mathcal{L}_A^l$ with $X$ such as:

$$\mathcal{L}_A^l [X] = AX$$

The components of $\mathcal{L}_A^l$ are given as:

$$\left( \mathcal{L}_A^l \right)_{ijkl} = A_{ik} I_{jl}$$

where $I$ is the identity tensor. Similarly, one can write the components of other fourth order operators for various other operations. In the following, the most commonly occurring operations are described.
Consider,

\[ L^A_X \{ X \} = X A \]

then,

\[ (L^A_X)^{ijkl} = A_{ij} I_{ik} \] (D.2)

When the transpose of the tensor \( X \) appears in the equation as:

\[ L^U_X \{ X \} = AX^T \]

then,

\[ (L^U_X)^{ij} = A_{ii} I_{kj} \] (D.3)

while the case,

\[ L^{st}_X \{ X \} = X^T A \]

results in,

\[ (L^{st}_X)^{ijkl} = A_{kj} I_{il} \] (D.4)

The \text{sym()} and \text{skew()} operators are also common. At first, let us consider the following transformation involving the symmetry operator.

\[ L^{sym}_C \{ X \} = \text{sym}(C \{ X \}) \]

then,

\[ (L^{sym}_C)^{ijkl} = \frac{1}{2}(C_{ijkl} + C_{jikl}) \] (D.5)

The skew operator is similarly written as:

\[ L^{skew}_C \{ X \} = \text{skew}(C \{ X \}) \]
from which one can conclude that,

$$\left( \mathcal{L}^{skew}_C \right)_{ijkl} = \frac{1}{2} (C_{ijkl} - C_{jikl})$$  \hspace{1cm} (D.6)

Finally, let us define

$$\mathcal{L}^{\text{trace}}_{C,A} [X] = \text{tr} (C [X]) A$$

with $C$ a fourth order tensor. With some algebraic manipulations, one can show that,

$$\left( \mathcal{L}^{\text{trace}}_{C,A} \right) = (A \otimes I) C$$  \hspace{1cm} (D.7)

As an example of application of the above transformations, let us consider equation (3.19) which expresses $\mathbf{U}^e$ in terms of $\mathbf{F}^e$. One can write this equation as follows:

$$\mathbf{U}^e = \mathbf{C} \left[ \mathbf{F}^e \right]$$  \hspace{1cm} (D.8)

where,

$$\mathbf{C} = \mathcal{L}^{\text{sym}}_p$$

$$\mathbf{P} = \mathcal{L}^i_B \mathcal{L}^{\text{sym}}_Q$$

$$\mathbf{Q} = \mathcal{L}^i_A$$

$$\mathbf{B} = (\mathbf{U}^e)^{-1}$$

$$\mathbf{A} = (\mathbf{F}^e)^T$$  \hspace{1cm} (D.9)

Thus, the relation between any two second order tensor can be written by forming relations as above.
Appendix E

Spline Basis Functions for the Die Design Problem

Let us define

\[ \eta = \frac{y - c}{l} \]

where \( c \) and \( l \) are defined in figure (4.2). The basis functions \( \phi_0(x, \eta) \) to \( \phi_6(x, \eta) \) for the family of dies considered in the numerical example, are then given as follows:

\[
\begin{align*}
\phi_0(x, \eta) &= x - C_1 \left( 1 - 270\eta^3 + 1215\eta^4 - 1458\eta^5 \right) \\
&\quad - C_2 \left( \eta - 54\eta^3 + 216\eta^4 - 243\eta^5 \right) \\
&\quad - C_3 \left( \frac{1}{2} \eta^2 - \frac{9}{2}\eta^3 + \frac{27}{2}\eta^4 - \frac{27}{2}\eta^5 \right) \text{ for } 0 \leq \eta < \frac{1}{3} \\
&= x \text{ for } \frac{1}{3} \leq \eta < \frac{2}{3} \\
&= x - C_4 \left( -512 + 3240\eta - 8100\eta^2 + 9990\eta^3 - 6075\eta^4 + 1458\eta^5 \right) \\
&\quad - C_5 \left( 80 - 512\eta + 1296\eta^2 - 1620\eta^3 + 999\eta^4 - 243\eta^5 \right)
\end{align*}
\]
\[- C_0 \left(-4 + 26\eta - 67\eta^2 + \frac{171}{2} \eta^3 - 54\eta^4 + \frac{27}{2} \eta^5\right) \]
for \(\frac{2}{3} \leq \eta \leq 1\)

\[
\phi_1(x, \eta) = -270\eta^3 + 1215\eta^4 - 1458\eta^5 \quad \text{for} \quad 0 \leq \eta < \frac{1}{3}
\]
\[
= -32 + 360\eta - 1620\eta^2 + 3510\eta^3 - 3645\eta^4 + 1458\eta^5 \quad \text{for} \quad \frac{1}{3} \leq \eta < \frac{2}{3}
\]
\[
= 0 \quad \text{for} \quad \frac{2}{3} \leq \eta \leq 1
\]

\[
\phi_2(x, \eta) = 36\eta^3 - 189\eta^4 + 243\eta^5 \quad \text{for} \quad 0 \leq \eta < \frac{1}{3}
\]
\[
= -\frac{16}{3} + 64\eta - 288\eta^2 + 612\eta^3 - 621\eta^4 + 243\eta^5 \quad \text{for} \quad \frac{1}{3} \leq \eta < \frac{2}{3}
\]
\[
= 0 \quad \text{for} \quad \frac{2}{3} \leq \eta \leq 1
\]

\[
\phi_3(x, \eta) = -\frac{3}{2} \eta^3 + 9\eta^4 - \frac{27}{2} \eta^5 \quad \text{for} \quad 0 \leq \eta < \frac{1}{3}
\]
\[
= -\frac{4}{9} + 14\eta - 19\eta^2 + \frac{65}{2} \eta^3 - 36\eta^4 + \frac{27}{5} \eta^5 \quad \text{for} \quad \frac{1}{3} \leq \eta < \frac{2}{3}
\]
\[
= 0 \quad \text{for} \quad \frac{2}{3} \leq \eta \leq 1
\]

\[
\phi_4(x, \eta) = 0 \quad \text{for} \quad 0 \leq \eta < \frac{1}{3}
\]
\[
= 31 - 360\eta + 1620\eta^2 - 3510\eta^3 + 3645\eta^4 - 1458\eta^5 \quad \text{for} \quad \frac{1}{3} \leq \eta < \frac{2}{3}
\]
\[
= -513 + 3240\eta - 8100\eta^2 + 9990\eta^3 - 6075\eta^4 + 1458\eta^5 \quad \text{for} \quad \frac{2}{3} \leq \eta \leq 1
\]

\[
\phi_5(x, \eta) = 0
\]
\[
\quad \text{for} \quad 0 \leq \eta < \frac{1}{3}
\]
\[
= -\frac{14}{3} + 55\eta - 252\eta^2 + 558\eta^3 - 594\eta^4 + 243\eta^5 \quad \text{for} \quad \frac{1}{3} \leq \eta < \frac{2}{3}
\]
\[
= -90 + 567\eta - 1401\eta^2 + 1710\eta^3 - 1026\eta^4 + 243\eta^5 \quad \text{for} \quad \frac{2}{3} \leq \eta \leq 1
\]

\[
\phi_6(x, \eta) = 0
\]
\[
\quad \text{for} \quad 0 \leq \eta < \frac{1}{3}
\]
\[
\begin{align*}
&= \frac{2}{9} - \frac{8}{3} \eta + \frac{25}{2} \eta^2 - \frac{57}{2} x^3 + \frac{63}{2} \eta^4 - \frac{27}{2} \eta^5 \quad \text{for} \quad \frac{1}{3} \leq \eta < \frac{2}{3} \\
&= -6 + 36 \eta - \frac{171}{2} \eta^2 + \frac{201}{2} \eta^3 - \frac{117}{2} \eta^4 + \frac{27}{2} \eta^5 \quad \text{for} \quad \frac{2}{3} \leq \eta \leq 1
\end{align*}
\]
Appendix F

The spline basis functions for the preform design problem

\[ \phi_1(\alpha) = 1 - 27\alpha^2 + 54\alpha^3 \text{ for } 0 \leq \alpha < \frac{1}{3} \]
\[ = 0 \text{ for } \frac{1}{3} \leq \alpha \leq 1 \]

\[ \phi_2(\alpha) = 27\alpha^2 - 54\alpha^3 \text{ for } 0 \leq \alpha < \frac{1}{3} \]
\[ = -4 + 36\alpha - 81\alpha^2 + 54\alpha^3 \text{ for } \frac{1}{3} \leq \alpha < \frac{2}{3} \]
\[ = 0 \text{ for } \frac{2}{3} \leq \alpha \leq 1 \]

\[ \phi_3(\alpha) = -3\alpha^2 + 9\alpha^3 \text{ for } 0 \leq \alpha < \frac{1}{3} \]
\[ = -\frac{4}{3} + 8\alpha - 15\alpha^2 + 9\alpha^3 \text{ for } \frac{1}{3} \leq \alpha < \frac{2}{3} \]
\[ = 0 \text{ for } \frac{2}{3} \leq \alpha \leq 1 \]

\[ \phi_4(\alpha) = 0 \text{ for } 0 \leq \alpha < \frac{1}{3} \]
\[ = 5 - 36\alpha + 81\alpha^2 - 54\alpha^3 \text{ for } \frac{1}{3} \leq \alpha < \frac{2}{3} \]
\[ -27 + 108\alpha - 135\alpha^2 + 54\alpha^3 \quad \text{for } \frac{2}{3} \leq \alpha \leq 1 \]

\[ \phi_5(\alpha) = 0 \quad \text{for } 0 \leq \alpha < \frac{1}{3} \]

\[ -\frac{2}{3} + 5\alpha - 12\alpha^2 + 9\alpha^3 \quad \text{for } \frac{1}{3} \leq \alpha < \frac{2}{3} \]

\[ -6 + 21\alpha - 24\alpha^2 + 9\alpha^3 \quad \text{for } \frac{2}{3} \leq \alpha \leq 1 \]

\[ \phi_6(\alpha) = 0 \quad \text{for } 0 \leq \alpha < \frac{2}{3} \]

\[ 28 - 108\alpha + 135\alpha^2 - 54\alpha^3 \quad \text{for } \frac{2}{3} \leq \alpha \leq 1 \]

\[ \phi_7(\alpha) = 0 \quad \text{for } 0 \leq \alpha < \frac{2}{3} \]

\[ -4 + 16\alpha - 21\alpha^2 + 9\alpha^3 \quad \text{for } \frac{2}{3} \leq \alpha \leq 1 \]
Bibliography


