METHOD FOR SOLVING INVERSE ELASTOVISCOPLASTIC PROBLEMS

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ABSTRACT: The solution of transient inverse elastoviscoplastic deformation problems is the topic of this paper. The inverse problems described here involve solving for boundary tractions and displacements given only approximate measurements of displacements at internal points to the body. A finite element formulation for the solution of these inverse problems, in conjunction with the regularization and future-time methods, is presented here. Several one-dimensional examples are given to demonstrate the method and to investigate the effect of various parameters on the solution.

INTRODUCTION

Inverse problems in solid mechanics arise when insufficient boundary conditions are prescribed, but additional information on the solution, such as displacements or strains at specified internal points in the body, is given. Then one must calculate the displacement, strain, and stress fields all over the body, including the boundary using the experimentally available data, which usually contain some amount of unspecified error.

Such problems arise in practice because an experimentalist can only approximately measure various quantities at discrete points in the body. Furthermore, these measurements are often performed at internal points in the body due, for example, to difficulties in placing sensors on boundaries where the specimen is in contact with another body. Inverse problems bring together the best features of both analysis and testing. Ideally, using known test data and approaching the problem from an inverse point of view can result in the actual determination of boundary and internal conditions, as well as several material properties.

Inverse problems by nature are ill-posed, and so their solution, which may not exist in a classical sense, does not satisfy the conditions of uniqueness and stability with small changes in the internal data, as has been described by Tikhonov and Arsenin (1977). In recent years, the main emphasis of inverse problems has been in transient heat conduction problems, for example, Beck (1970), Zabaras and Liu (1988), and Zabaras and Ruan (1989).

An analysis of inverse elasticity problems has been performed earlier by Maniatty et al. (1989). In this reference, the boundary tractions and displacements in an elastically deformed two-dimensional body are calculated using approximate information on displacements and strains at a finite number of internal points in the body. The natural extension of the inverse techniques presented in Maniatty et al. (1989) is to consider inverse deformation problems with material nonlinearities. Such problems, which are very common in practice, will be the subject of this paper. In addition to the com-
plications associated with inverse elasticity problems, one has to account here for the history dependence of the deformation. Since one-dimensional applications are reported herein, dynamic effects have been included to guarantee that the problem of calculating boundary conditions from specified internal displacement data is indeed an inverse one. Such dynamic nonlinear material deformation problems are time dependent explicitly through the inertia effect, as well as implicitly through the deformation history of the material.

This paper will start by defining a general inverse, transient, nonlinear material deformation problem. More specifically, the problem of calculating the time history of the traction and displacement boundary conditions from internal displacement history data of a specimen undergoing small elastoviscoplastic deformation will be analyzed. Both the regularization technique and the future-time step technique are employed here. The future-time technique is similar to the function specification technique applied in the area of inverse heat conduction and described earlier by Eeck et al. (1985). Various one-dimensional examples will be given to demonstrate the effect on the solution of the error in the displacement data, the location of the sensors, the time step used, and the amount of future displacement information used. Finally, conclusions and future applications of the presented methods will be given.

**DEFINITION OF GENERAL PROBLEM**

Consider a three-dimensional body $B$ with boundary $\partial B$ and let $\sigma_{ij}(x,t)$, $\varepsilon_{ij}(x,t)$, and $u_i(x,t)$, $i = 1, 3, j = 1, 3$, denote the stress, strain, and displacement fields, respectively, where $x$ defines the location of a point in the body and $t$ denotes time. The body $B$ with boundary $\partial B$ is assumed to be homogeneous and isotropic with its elastic strains calculated from Hooke’s law, while the nonelastic strain rates are specified from a given elastoviscoplastic constitutive law.

Following the familiar notation of Fung (1965), the governing equilibrium equations, neglecting body forces, and the traction and displacement boundary conditions are as follows

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad \text{on } B \quad (1)$$

$$\sigma_{ij} n_j = \tau_i \quad \text{on } \partial B_{1i} \quad (2)$$

$$u_i = \Delta_i \quad \text{on } \partial B_{12} \quad (3)$$

where $\rho$ is the density; $\ddot{u}_i$ is the acceleration, $n(x)$ is the unit normal at $x \in \partial B$, $\partial B = \partial B_{1i} \cup \partial B_{12}$, and $\tau_i$ and $\Delta_i$ are the boundary tractions and displacements. In addition, the symbol $\dot{}$ indicates a spatial derivative and $\ddot{}$ indicates a time derivative. All variables are functions of location $x$ and time $t$.

In addition to the preceding equations, assuming small strains and rotations, one can write the following familiar tensor notation

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (4)$$

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^n \quad (5)$$

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\[ \sigma_{ij} = \lambda \varepsilon_{ii} \delta_{ij} + 2\mu \varepsilon_{ij} \]  
\[ \ddot{\varepsilon}_{ij} = f_{ij}(\sigma_{rs}, q_{kl}) \]  
\[ \dot{q}_{kl} = g_{kl}(\sigma_{ij}, q_{rs}) \]  

where \( q_{kl} \) are state variables usually representing micromechanical parameters such as hardness, dislocation density, etc. Eq. 4 defines the strains from the displacements and Eq. 5 decomposes the total strains into the elastic (indicated with superscript \( e \)) and nonelastic (indicated with superscript \( n \)) ones. Eq. 6 expresses Hooke's law with \( \lambda \) and \( \mu \) being the Lamé constants and Eqs. 7 and 8 define the general form of an elastoviscoplastic constitutive model with state variables, for example, that proposed by Hart et al. (1976) and Hart (1976) for the nonelastic strain rate, \( \dot{\varepsilon}_{ij}^n \), where the nonelastic strains are assumed to be incompressible (\( \varepsilon_{ii}^n = 0 \)). The nonelastic strain rates are generally assumed to be functions only of the current values of the stresses \( \sigma_{ij} \) and state variables \( q_{kl} \), which follow a specified evolution law given in the form of Eq. 8. For a direct deformation problem, the following boundary value problem can be defined. Given \( \tau_i \) and \( \Delta_i \), solve Eqs. 1–8 for the stress, strain, and displacement fields. It is obvious that the residual strains/stresses at the end of the deformation process will be part of the solution. The solution of such problems using the finite element method has been very well established and the reader is referred to the excellent texts by Zienkiewicz (1979) and Bathe (1982).

In this paper, the goal is to calculate the boundary tractions \( \tau_i \) and displacements \( \Delta_i \) given approximate information on displacements at some points \( x \) internal to the body \( B (x \in B, x \notin \partial B) \). That is, given the following
\[ u_i(x_k, t) \equiv \ddot{u}_{ik}(t) \quad x_k \in B, x_k \notin \partial B, k = 1, M \]  

where \( M \) is the number of internal measurement points (sensor locations), solve Eqs. 1 and 4–9 for the boundary tractions and displacements. Obviously, certain additional restrictions should be imposed on the number of sensors \( M \), their locations, and the percentage of error in the given data such that a physically meaningful solution can be obtained. In the method to be reported next, the boundary tractions are considered to be the primary unknowns of the problem since they are more sensitive to errors in the internal data. Naturally, with known tractions one can, by solving a direct problem, calculate the boundary displacements and various internal quantities.

**Solution Procedure**

A finite element approach is followed. The region \( B \) is discretized into finite elements in such a way that sensors are located at finite element nodes. The number of sensors is generally much less than the number of finite element nodes. A weak Galerkin formulation of Eqs. 1–3 is involved, which finally, in discretized form, gives the following system of equations (Owen and Hinton 1986)
\[ [\mathbf{M}][\ddot{\mathbf{u}}] + [\mathbf{K}][\mathbf{u}] = [\mathbf{F}] \]  
where \( \{\mathbf{u}\} \) is the nodal displacement vector, \( \{\ddot{\mathbf{u}}\} \) is the nodal acceleration vector, \( [\mathbf{M}] \) is the mass matrix, \( [\mathbf{K}] \) is the stiffness matrix, and \( [\mathbf{F}] \) is the vector of nodal forces, which, in addition to terms associated with the boundary
tractions, contains the pseudoloads created by the viscoplastic effects (see Appendix I for a development of this system). Following a time stepping procedure, with \( t_n \) being the time at step \( n \), \( n = 1, 2, \ldots, N \), and \( \Delta t \) the value of the constant time step, a trapezoidal type of time marching scheme for Eq. 10 takes the form

\[
[M] \frac{[u_{n+1} - 2u_n + u_{n-1}]}{(\Delta t)^2} + [K]((1 - \theta)u_n + \theta u_{n+1})
\]

\[= (1 - \theta)[F_n] + \theta[F_{n+1}] \] ................................................................. (11)

with \( 0.5 \leq \theta \leq 1.0 \), and where the displacement, velocity, and acceleration fields are considered known at \( t \leq t_n \), and so \( \{u_{n+1}\} \equiv \{u(t_{n+1})\} \) is assumed to be the basic unknown of the problem. It is clear from Appendix I that Eqs. 10 and 11 were obtained by assuming an explicit (Euler) integration scheme for the nonelastic strain rates. Consequently, Eq. 11 is only conditionally stable (Ortiz and Popov 1985). The load vector \( \{F_{n+1}\} \) is a function of the unknown boundary traction vector \( \{\tau_{n+1}\} \equiv \{\tau(t_{n+1})\} \), therefore, Eq. 11 cannot be used as it is to solve for \( \{u_{n+1}\} \). Note that part of \( \{u_{n+1}\} \) is approximately known through Eq. 9. Let \( \{u_{n+1}^*\} \) be the part of \( \{u_{n+1}\} \) at the finite element nodes that coincide with the sensor locations. Then

\[
\{u_{n+1}^*\} = [Q]\{u_{n+1}\} \] ........................................................................... (12)

where \([Q]\) is a proper rectangular transformation matrix. The goal is to select the traction \( \{\tau_{n+1}\} \) in such a way that \( \{u_{n+1}^*\} \) is closest to \( \{u_{n+1}\} \equiv \{u(t_{n+1})\} \), where \( \{u_{n+1}\} \) is the vector of experimentally determined displacements (Eq. 9). If an iterative scheme is developed with a truncated Taylor series expansion of \( \{u_{n+1}\} \) around \( \{u_{n+1}^*\} \), given by

\[
\{u_n\} = \{u_{n+1}^*\} + [X_1]\{\tau_{n+1} - \tau_{n+1}^{-1}\} \] .................................................. (13)

where \( r \) is the iteration number and \([X_1]\) is the sensitivity coefficient matrix defined as

\[
[X_1] = \left[ \begin{array}{c} \frac{\partial \{u_n\}}{\partial \{\tau_{n+1}\}} \\ \frac{\partial \{\tau_{n+1}\}}{\partial \{\tau_{n+1}\}} \end{array} \right] \mathrm{evaluated\ at\ } \{\tau\} = \{\tau_{n+1}^{-1}\} \] ........................................ (14)

then a first goal can be to minimize the following function of \( \{\tau_{n+1}^r\} \)

\[
E_1 = \{u_{n+1}^r - \hat{u}_{n+1}\}^T [X_1]\{ \tau_{n+1}^r - \tau_{n+1}^{-1} \} \]

Substituting Eqs. 12 and 13 into 15 and taking \( \partial E_1 / \partial \{\tau_{n+1}^r\} = 0 \), a value for \( \{\tau_{n+1}^r\} \) can be found. Then this updated traction can be used to solve the direct problem for \( \{u_{n+1}^r\} \), and the process continues until convergence is achieved at each time step. As it is known from previous inverse problems, this procedure leads to a singular system of equations, so some other technique must be developed.

An alternative procedure is to minimize the error between measured and calculated displacements not only at \( t_{n+1} \), but on the time interval \( t_{n+1} \leq t \leq t_{n+p} \), where \( p \geq 2 \). Then the main unknowns of the problem become the tractions \( \{\tau_{n+1}\}, \{\tau_{n+2}\}, \ldots, \{\tau_{n+p}\} \), which are calculated together at once rather than sequentially. Then the iterative scheme becomes

\[
\{u_{n+p}^i\} = \{u_{n+i}^{-1}\} + \sum_{j=1}^i [X_0]\{\tau_{n+j}^r - \tau_{n+j}^{-1}\} \quad i = 1, p \] ...................................................... (16)
where

\[ [X_j] = \left[ \frac{\partial \{u_{n+i}\}}{\partial \{\tau_{n+i}\}} \right] \text{ evaluated at } \{\tau_{n+j}\} = \{\tau_{n+j}^{-1}\} \quad \text{for } j = 1, 2, \ldots, i \]

\[ [X_j] = 0 \quad \text{for } j > i \]

(17)

since the displacements \( \{u_{n+i}\} \) are only functions of the current and previous tractions. The error to be minimized with respect to all \( \{\tau_{n+j}\}, j = 1, p \) is

\[ E_2 = \sum_{i=1}^{p} \left( (u_{n+i}^r - \hat{u}_{n+i})^T (u_{n+i}^r - \hat{u}_{n+i}) \right) \]

(18)

Note that with this formulation, the number of unknown tractions has been increased considerably. This method also pays no special attention to the ill-posedness of the problem and has found to be unstable in addition to being very difficult to implement computationally, especially in multidimensions.

Since it is known that the tractions must have a certain smoothness over short time periods, it can be temporarily assumed at each iteration that the traction \( \{\tau\} \) varies linearly on the time interval \( t_n \leq t \leq t_{n+p}, \) i.e.,

\[ \{\tau_{n+i}\} = \frac{t_{n+i} - t_n}{\Delta t} \{\tau_{n+1} - \tau_n\} + \{\tau_n\} \quad i = 1, p \]

(19)

Therefore, since the traction is known at time \( t_n, \) by assuming temporarily a traction \( t_{n+1}, \) all the tractions for the times \( t_{n+i}, i = 1, p \) can be found and used in the minimization of the function \( E_2, \) which now has \( \{\tau_{n+i}\} \) as the only independent variable. These and similar ideas have been previously applied in inverse heat conduction problems by Beck (1970), Zabaras and Liu (1988), and Zabaras and Ruan (1989). So the goal becomes to minimize with respect to the only independent variable \( \{\tau_{n+1}\}, \) the following

\[ E_3 = \sum_{i=1}^{p} \left( (u_{n+i}^{r-1} - \hat{u}_{n+i})^T (u_{n+i}^{r-1} - \hat{u}_{n+i}) \right) + \left( \tau_{n+1}^{r-1} - \tau_{n+1}^{-1} \right)^T [R] \left( \tau_{n+1}^{r-1} - \tau_{n+1}^{-1} \right) \]

(20)

where

\[ \{u_{n+i}^{r-1}\} = \{u_{n+i}^{r-1}\} + [Z_i] \{\tau_{n+1}^{r-1} - \tau_{n+1}^{-1}\} \]

(21a)

with

\[ [Z_i] = \left[ \frac{\partial \{u_{n+i}^{r}\}}{\partial \{\tau_{n+1}^{r}\}} \right] \text{ evaluated at } \{\tau_{n+1}\} = \{\tau_{n+1}^{-1}\} \]

(21b)

and where \([R]\) is a positive definite regularization matrix that controls the relative magnitude of the tractions between iterations and is included to further stabilize the solution (Tikhonov and Arsenin, 1977). It is obvious that \( E_3, \) which includes future time information, is used to calculate current values \( t = t_{n+i} \) of the traction \( \{\tau\}. \) After performing this minimization, the final equation obtained is

\[ \sum_{i=1}^{p} ([Z_i]^T (\hat{u}_{n+i} - u_{n+i}^{r-1})) = \left[ \sum_{i=1}^{p} ([Z_i]^T [Z_i]) + [R] \right] \{\tau_{n+1}^{r-1} - \tau_{n+1}^{-1}\} \]

(22)
The sensitivity coefficients \([Z_i]\) relate the effect of the traction being sought at the \(n + 1\) time step to the displacements at the sensor locations at the \(n + i, i = 1, p\) time steps. These displacements are functions of the traction at the \(n + 1\) time step and of the tractions at the \(n + i, i = 2, p\) time steps, which are in turn also functions of the traction at the \(n + 1\) time step through Eq. 19. The sensitivity coefficients are found implicitly using a finite difference approximation, which symbolically can be represented as follows.

\[
\left[ \frac{\partial \{\mathbf{u}^*_n\}}{\partial \{\mathbf{\tau}_n\}} \right]_{i+1} = \frac{\{\mathbf{u}^*_n\} - \{\mathbf{u}^*_n\}}{(1 + \epsilon)\{\mathbf{\tau}_n\} - \{\mathbf{\tau}_n\}} = \frac{\{\mathbf{u}^*_n\} - \{\mathbf{u}^*_n\}}{\epsilon\{\mathbf{\tau}_n\}} \quad \text{.................................. (23)}
\]

where \(\epsilon\) is a very small number (in the computations reported later \(\epsilon = 0.00001\)) and \(\{\mathbf{u}^*_n\}\) is the displacement vector that results when the traction vector \((1 + \epsilon)\{\mathbf{\tau}_n\}\) is applied at time \(t = t_{n+1}\). Note that in calculating the sensitivity coefficients the traction constraint is being used. Using Eqs. 22 and 23, one can update the traction to a better value for the next iteration. This scheme proves to be stable.

In summary, the algorithm proceeds as follows:

1. Assume a traction vector at the iteration \(r = 0, \{\mathbf{\tau}_{n-1}\}\), to be a small number or a better estimate if available.
2. Let \(r = r + 1\). Calculate the tractions \(\{\mathbf{\tau}_{n+1}\}\), \(i = 1, p\), based on the assumed tractions \(\{\mathbf{\tau}_{n}^{r-1}\}\) and the known tractions \(\{\mathbf{\tau}_{n}\}\) using Eq. 19.
3. Solve the direct problem (Eq. 11) using these tractions to get \(\{\mathbf{u}_{n+1}^{r-1}\}\), \(i = 1, p\).
4. Compute the updated tractions \(\{\mathbf{\tau}_{n+1}^r\}\) from Eqs. 22 and 23.
5. Check for convergence on the tractions. If the root mean square norm between the tractions at the \(r\) and \(r - 1\) iterations is less than a given tolerance, then convergence has been reached.
6. If the convergence has been attained, then begin the next time step; otherwise use the updated tractions as the new assumed tractions and continue with step 2.

**Examples**

A one-dimensional elastoviscoplastic bar with a dynamic loading, as shown in Fig. 1, is used for all of the examples presented here. The material follows a very simple elastoviscoplastic model where

\[
\dot{\epsilon}^n = \gamma (\sigma - \sigma_Y - H' \epsilon^n) \quad \text{for} \quad |\sigma| \geq |\sigma_Y + H' \epsilon^n| \quad \text{................................. (24a)}
\]

\[
\dot{\epsilon}^n = 0 \quad \text{for} \quad |\sigma| < |\sigma_Y + H' \epsilon^n| \quad \text{................................. (24b)}
\]

where \(\gamma\) is a fluidity parameter taken as 0.01, \(\sigma_Y\) is the uniaxial yield stress, which is taken as 10, and \(H'\) is a strain hardening parameter, which is taken as 5,000. In addition, the elastic modulus is taken as 10,000, and the mass of the bar is 5 and assumed to be evenly distributed. Finally, all quantities are assumed to be in consistent units.

A direct problem was first solved to generate the simulated experimental data. Five linear elements were used and the node closest to the boundary where the traction is being applied \((x = 0.875)\) was taken as the “sensor”
location. The loading condition and resulting displacements at the assumed "sensor" location are shown in Fig. 2. The inertial and viscous effects can be seen here as the displacement lags the loading and some oscillation continues to occur at the end of the loading. Using future information in the formulation, as outlined earlier, becomes particularly important here since the displacement response at the sensor location lags the applied traction, making it necessary to use some future displacement information in order to obtain the traction at the given time increment. A similar phenomena has been observed and studied in transient heat conduction problems where the internal temperature response lags the surface heat flux condition (Beck et al. 1985).

In order to simulate random error, the displacements were multiplied by $(1 + 0.02w)$, where $w$ is a random number on the interval $[-1, 1]$. Note that this error is an additional one to the relatively small error produced by solv-

FIG. 2. Applied Traction at $x = 1$ and Resulting Displacement at Sensor Located at $x = 0.875$ for Elastoviscoplastic Bar Shown in Fig. 1
FIG. 3. Predicted Traction Using Least Squares Inverse Technique \((p = 1)\) with 2% Random Error in Displacement Data \((\Delta t = 0.01, \text{ Sensor Located at } x = 0.875)\)

FIG. 4. Predicted Traction Using Inverse Technique for Same Conditions as in Fig. 3, but Using One Future Time Step, i.e., \(p = 2\), with 2% Random Error in Given Displacements \((\Delta t = 0.01, \text{ Sensor Located at } x = 0.875)\)

The direct FEM algorithm. The time step was taken as \(\Delta t = 0.01\). Using a simple least squares minimization \((p = 1)\) in each time step, an unstable solution was obtained as can be seen in Fig. 3. If one future-time step is used, i.e., \(p = 2\) in Eq. 18, without regularization, a very accurate solution is obtained as shown in Fig. 4. Regularization was found not to be necessary and one future time step was found to be all that was necessary to stabilize.
the problem in this case. From the computed boundary condition, the plastic strains were calculated a long time after the loading stabilized to a constant value and are plotted in Fig. 5. The solution was also observed for different values of random error by multiplying the displacements calculated from a direct FEM algorithm by \((1 + 0.05w)\) and \((1 + 0.08w)\) where \(w\) is the same as before and these results are plotted in Figs. 6 and 7, respectively. Again,

**FIG. 5.** Inversely Predicted Plastic Strains Long Time After Loading Stabilized to Fixed Value for Traction Solution Shown in Fig. 4

**FIG. 6.** Predicted Traction Using Inverse Technique with One Future Time Step \((p = 2)\) with 5% Random Error in Given Displacements \((\Delta t = 0.01, \text{ Sensor Located at } x = 0.875)\)
FIG. 7. Predicted Traction Using Inverse Technique with One Future Time Step $(p = 2)$ with 8\% Random Error in Given Displacements ($\Delta t = 0.01$, Sensor Located at $x = 0.875$)

only one future-time step was used without regularization. As it can be seen, even with 8\% random error, a rather accurate solution was obtained. The use of additional “sensors” at locations $x < 0.875$ was also investigated and was found to have no noticeable effect on the solution. The “sensor” nearest the boundary is of primary importance, while the additional “sensors” fur-

FIG. 8. Predicted Traction Using Inverse Technique with One Future Time Step $(p = 2)$ for Same Conditions as Used in Fig. 4, but with Reduced Time Step, 2\% Random Error is in Given Displacements ($\Delta t = 0.005$, Sensor Located at $x = 0.875$)
FIG. 9. Predicted Traction Using Inverse Technique with Two Future Time Steps ($p = 3$) and Regularization for Same Conditions as Used in Fig. 8, 2% Random Error is in Given Displacements ($\Delta t = 0.005$, Sensor Located at $x = 0.875$)

FIG. 10. Predicted Traction Using Inverse Technique with One Future Time Step ($p = 2$) and Regularization for Same Conditions as Used in Fig. 8 with 2% Random Error in Given Displacements, but with Sensor Further Removed from Boundary, ($\Delta t = 0.005$, Sensor Located at $x = 0.5$)

Further from the boundary do not provide significant new information on the solution.

The effect of the time step was also examined by solving the same problem with $\Delta t = 0.005$, again with 2% random error added to the displacement.
data in the same manner as discussed earlier. The results with one future time step ($p = 2$) are shown in Fig. 8. Reducing the time step has reduced the stability of the solution, which is expected due to the lag between the internal displacement and the applied traction. The results are stabilized by increasing the number of future-time steps to two ($p = 3$), which is shown in Fig. 9. In this case, regularization was required for the solution to converge. A simple diagonal regularization matrix with a constant normalized value of 0.06 along the diagonal was used.

The effect of the location of the “sensor” was also investigated. The same problem was solved, again taking $\Delta t = 0.005$, with the “sensor” located at $x = 0.5$. The solution with one future-time step and regularization is shown in Fig. 10. As it can be seen, the solution is very unstable, largely due to the time lag between the internal displacement and the boundary traction, which has become more pronounced with the “sensor” further from the applied traction. By increasing the number of future-time steps to two, the solution improves substantially as can be seen in Fig. 11. In this case, regularization was again necessary for the solution to converge. The regularization matrix used was a diagonal matrix with a constant normalized value of 0.01 along the diagonal.

**Conclusions**

A general finite element formulation based on regularization and the future-time method has been proposed for the solution of a class of inverse deformation problems with material nonlinearities. The current method has been shown to produce good results for the solution of one-dimensional transient inverse deformation problems where an elastoviscoplastic constitutive model is used to describe the inelastic material behavior. Using future-time
information and regularization in the formulation was found to be an effective and easy way to obtain a stable solution. Also, the solution was found to be fairly insensitive to the amount of random error introduced into the displacement data. The “sensor” location, on the other hand, was found to have a more pronounced effect on the boundary results. More specifically, the boundary tractions deteriorate as the “sensor” is moved further from the boundary, but this effect can be overcome by increasing the number of future time steps used in the formulation. This is due to the lag between the internal displacements and the applied tractions. The boundary solution was also found to become less stable as the time step became smaller, which is also related to the time lag between the internal displacements and the traction and can be compensated for by increasing the number of future time steps used and by introducing regularization.

The calculations discussed herein can be quite extensive and expensive in multidimensional situations. The main time-consuming calculations include the evaluation of the sensitivity coefficients as well as the solution of the direct problem during the iterative process. Some of these calculations can be avoided by keeping the sensitivity coefficients constant for a small period of time and by introducing spatial regularization with a reduced number of boundary unknowns. Such computational issues are currently under investigation for two-dimensional problems and they will be further discussed in a later publication.

Potential applications of the methodology given here to the nondestructive evaluation of residual stresses and material parameters have already been addressed by Maniatty and Zabaras (1988). Finally, a statistical analysis and an estimation of confidence bounds on the solution will be a future goal of this ongoing research in inverse problems relating experimental and numerical analysis of thermomechanical processes.

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APPENDIX I. FINITE ELEMENT FORMULATION FOR TRANSIENT ELASTOVISCOPLASTIC PROBLEMS

Following the notation of Owen and Hinton (1986), and using the principle of virtual work at time \( t_n \), in the absence of body forces, one can write

\[
\int_B \{\delta \epsilon_n\}^T \{\sigma_n\} d\Gamma + \int_B \{\delta \mathbf{u}_n\}^T \{\mathbf{p}_n\} d\Gamma - \int_{\partial B} \{\delta \mathbf{u}_n\}^T \{\mathbf{\tau}_n\} d\Gamma = 0 \quad \ldots \quad (25)
\]

where the symbol \( \delta \) indicates variation. Discretizing this into finite element form gives

\[
[M_n][\ddot{\mathbf{u}}_n] + \{\mathbf{p}_n\} = \{\mathbf{f}_n\} \quad \ldots \quad (26)
\]

where \([M_n]\) is the mass matrix, \( \{\ddot{\mathbf{u}}_n\} \) is the vector of nodal accelerations, \( \{\mathbf{p}_n\} \) is the vector of internal resisting forces, and \( \{\mathbf{f}_n\} \) is the load vector that results
from the applied boundary tractions. The mass matrix is defined by assembling the element mass matrices \((e = 1, N_1)\) defined as

\[
M_{e}^{c}_{\alpha/\beta} = \int_{B_e} \rho_o \delta_{ij} \Psi_\alpha^c \Psi_\beta^c dB_e \tag{27}
\]

and the load vector by assembling the element force vectors \(f_{e}^c (e = 1, N_2)\) given as

\[
f_{e}^c = \int_{\partial B_{e}} \tau_{i}(t_n) \Psi_\alpha^c d\Gamma_e \tag{28}
\]

where \(N_1\) is the number of elements, while \(N_2\) is the number of linear boundary segments; \(\delta_{ij}\) is the Kronecker delta, and \(\Psi_\alpha^c\) and \(\Psi_\beta^c\) indicate element shape functions; for linear elements in one dimension, \(i = j = 1\), \(\alpha, \beta = 1, 2\). In this work, a lumped mass matrix was used.

The internal resisting forces in Eq. 26 can be written as a function of the stresses as

\[
\{p_n\} = \int_{B} [B]^T \{\sigma_n\} dB \tag{29}
\]

where \([B]\) is defined such that \(\{\epsilon_n\} = [B] \{u_n\}\), i.e., \([B]\) reveals the strains from the displacements. The stress \(\{\sigma_n\}\) is a function of the element strains \(\{\epsilon_n^e\}\) through Hooke’s Law, Eq. 6; from this the following relation can be obtained

\[
\{\sigma_n\} = [D] \{\epsilon_n^e\} \tag{30}
\]

where \([D]\) is the elastic constants matrix. Substituting Eqs. 5 and 30 into Eq. 29, the resulting equation is

\[
\{p_n\} = \int_{B} [B]^T [D] [B] dB \{u_n\} - \int_{B} [B]^T [D] \{\epsilon_n^e\} dB \tag{31}
\]

where the nonelastic strains \(\{\epsilon_n^e\} = \{\epsilon^e(t_n)\}\) are calculated as

\[
\{\epsilon_n^e\} = \{\epsilon_{n-1}^e\} + \{\dot{\epsilon}_{n-1}^e\} \Delta t_n \tag{32}
\]

with the time step \(\Delta t_n = t_n - t_{n-1}\). The first term in Eq. 31 is the elastic component of the internal resisting forces and the second term is the nonelastic (viscoplastic) component. Eq. 31 can then be rewritten as

\[
\{p_n\} = [K] \{u_n\} - \{V_n^p\} \tag{33}
\]

where \([K]\) is the stiffness matrix and is the same as in elasticity, and \(\{V_n^p\}\) is the nonelastic internal resisting force, which can be considered as a pseudoload.

Substituting Eq. 33 into 26, the final equation that must be satisfied at each time increment is

\[
[M_n] \{\dot{u}_n\} + [K] \{u_n\} = \{F_n\} \tag{34}
\]

where \(\{F_n\}\) combines both the load vector due to the tractions \(\{f_n\}\) and the pseudoload due to the internal nonelastic resisting forces \(\{V_n^p\}\).
**APPENDIX II. REFERENCES**


**APPENDIX III. NOTATION**

The following symbols are used in this paper:

\[
\begin{align*}
B & = \text{domain of the body being considered;} \\
[B] & = \text{matrix that reveals strains from displacements;} \\
\partial B & = \text{boundary of } B; \\
\partial B_1 & = \text{part of } \partial B \text{ with a traction boundary condition;} \\
\partial B_2 & = \text{part of } \partial B \text{ with a displacement boundary condition;} \\
E & = \text{error to be minimized;} \\
\{F\} & = \text{load vector due to traction boundary condition and viscoplastic pseudoloads;} \\
\{f_n\} & = \text{load vector due to traction boundary condition;} \\
[K] & = \text{elastic stiffness matrix;} \\
M & = \text{number of internal measurement points;} \\
[M] & = \text{mass matrix;} \\
n & = \text{unit normal to } \partial B; \\
N_1 & = \text{number of elements;} \\
N_2 & = \text{number of linear boundary segments;} \\
p & = \text{number of future time steps plus one;}
\end{align*}
\]
\{p_n\} = \text{vector of internal resisting forces;}
[Q] = \text{transformation matrix that reveals the displacements at the measurement points from the complete vector of displacements;}

q = \text{state variables;}
\dot{q} = \text{state variable rate;}

[R] = \text{regularization matrix that controls the relative difference of the tractions between iterations;}

\tau = \text{time;}
\{u\} = \text{displacement vector;}
\{\dot{u}\} = \text{measured internal displacements;}
\{\ddot{u}\} = \text{acceleration vector;}

\{u^*\} = \text{calculated displacements at locations of internal measurements;}
\{V^*_n\} = \text{nonelastic internal resisting forces;}

[X] = \text{sensitivity coefficient matrix } \equiv [\partial \{u\} / \partial \{\tau\}];

x = \text{vector locating a point in } B;

[Z] = \text{sensitivity coefficient matrix corresponding to sensor locations } = [Q][X];

\Delta \tau = \text{prescribed displacements on } \partial B_{12};

\Delta t = \text{time step;}

\epsilon = \text{strains;}

\epsilon^e = \text{elastic strains;}

\epsilon^n = \text{nonelastic strains;}

\dot{\epsilon}^n = \text{nonelastic strain rate;}

\lambda = \text{Lamé parameter;}

\mu = \text{Lamé parameter;}

\Psi = \text{element shape function;}

\rho = \text{density;}

\sigma = \text{stresses;}

\{\tau\} = \text{traction vector on } \partial B_{12} \text{ that is being sought; and}

\theta = \text{parameter used in time marching scheme.}