A COMPUTATIONAL MODEL FOR THE FINITE ELEMENT ANALYSIS OF THERMOPLASTICITY COUPLED WITH DUCTILE DAMAGE AT FINITE STRAINS

AKKARAM SRIKANTH AND NICHOLAS ZABARAS *

Sibley School of Mechanical and Aerospace Engineering, 188 Frank H. T. Rhodes Hall, Cornell University, Ithaca, NY 14853-3801, U.S.A.

SUMMARY

An updated Lagrangian implicit FEM model for the analysis of large thermo-mechanically coupled hyperelastic-viscoplastic deformations of isotropic porous materials is considered. An appropriate framework for constitutive modelling is introduced that includes a stress-free thermally expanded configuration and a plastically deformed unstressed damaged configuration. A two-level iterative scheme is employed at each time increment to solve the field equations governing the conservation of momentum (mechanical step) and the conservation of energy (thermal step) for the coupled thermo-mechanical problem. Exact linearizations for the calculation of the tangent stiffness are performed in each of these solution steps. A fully implicit, thermo-mechanically coupled and incrementally objective Euler-backward radial return based map is developed for the time integration of the constitutive equations. The present model is used to analyse a number of benchmark examples including metal forming processes wherein temperature and the accumulated damage play an important role in influencing the deformation mechanism and the nature of the deformed workpiece. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: thermoplasticity; damage; forming; finite deformations

1. INTRODUCTION

In this study, a computational model for the analysis of finite inelastic deformations is presented, using an updated Lagrangian formulation which takes into account the influence of temperature and damage. The nature of coupling between the thermal field and the deformation field is as follows. The thermal field influences the mechanical field via the effect of thermal expansion and through the temperature dependence of the mechanical material response. In addition, the mechanical field influences the thermal field through dissipative structural plastic heating as well as by deformation induced changes in the workpiece geometry, porosity and in the nature of the applied thermal boundary conditions.

Typically, forming processes are accompanied by notable damage accumulation causing internal deterioration which may lead to the occurrence of macroscopic failure. An important issue in the
design and analysis of forming processes is the prediction of ductile fracture which is used to model workability. The dramatic influence of temperature on the localization of deformation into shear bands in a ductile fracture process has been demonstrated by Tvergaard and Needleman [1]. Thus, in order to perform realistic simulations of metal forming processes it is necessary to take into account the effects of temperature as well as internal damage.

In this paper, a constitutive framework is introduced in which the multiplicative decomposition of the deformation gradient includes a deformation induced by unconstrained thermal expansion as well as a plastic deformation gradient produced by slip and from macroscopic volume changes resulting from damage. The notion of an intermediate thermally expanded unstressed configuration is introduced in this context. Alternative formulations for thermoplasticity have been proposed among others by Lush [2] and by Simo and Miehe [3]. The scheme proposed by Lush uses a hypo-elastic model and an additive decomposition of the strain rate. Lush describes an efficient, implicit (Euler-backward), incrementally objective time integration scheme for rate-dependent isotropic constitutive models. Simo and Miehe proposed a thermoplasticity model using the principle of maximum dissipation within the framework of the multiplicative deformation gradient decomposition for rate independent thermo-elasto-plasticity. They also proposed an alternative solution scheme based on a global product formula algorithm constructed via an operator split of the coupled thermo-mechanical field equations.

The prediction of material processing defects using a damage mechanics approach is an active research area [4–7]. In an approach to modelling isotropic damage, ductile failure by the nucleation, growth and coalescence of voids is modelled in terms of a single internal variable representing the volume void fraction. The classical damage model due to Gurson [8] follows this approach and has been extensively used to model elasto-plastic damage at isothermal conditions. A notable contribution in this regard is the work by Aravas where an Euler backward integration algorithm for pressure-dependent infinitesimal elasto-plasticity for a Gurson-based model was proposed [9]. The Gurson model has been modified by Tvergaard and Needleman to account for rate sensitivity, void coalescence at failure, modelling periodic arrays of voids, etc. The version of the model presented in [1] is used here.

The purpose of this paper is to couple such a ductile damage model with the proposed thermo-plasticity model within a multiplicative elasto-viscoplasticity framework. A novel updated Lagrangian formulation for the large deformation of isotropic, thermo-hyperelastic-viscoplastic porous materials is presented. A fully implicit and fully thermo-mechanically coupled exponential map constitutive time integration procedure for the Gurson–Tvergaard–Needleman damage model is developed. The proposed model enables the study of material defects in both cold-working and hot-working regimes where the rate sensitivity effects are pronounced. The stress and strain measures selected for the present analysis are the same as those advocated by Anand and colleagues for isotropic plasticity [10] and the integration scheme proposed here is an extension of the work by Weber and Anand [11] where the deformation of isotropic hyperelastic viscoplastic materials was examined with a multiplicative plasticity framework and using a logarithmic strain measure. A Newton–Raphson method is used in the present work to solve the coupled thermo-mechanical equations and exact linearizations of the field and constitutive equations are developed in order to compute the tangent stiffness and the consistent material linearized moduli, respectively. The importance of such linearizations has been emphasized elsewhere [12, 13]. Various computational features of the present implementation are also described. Finally, we present a comprehensive set of representative numerical examples including some in the simulation of metal forming processes that demonstrate the performance of the proposed computational scheme in handling the
2. A CONSTITUTIVE FRAMEWORK FOR THERMO-MECHANICAL INELASTIC ANALYSIS WITH DAMAGE

Consider the smooth deformation mapping defined as $\mathbf{x} = \phi(\mathbf{X}, t) : \mathbb{B}_0 \rightarrow \mathbb{R}^3$ which maps particles $\mathbf{X}$ in the reference configuration $\mathbb{B}_0$ to their positions $\mathbf{x}$ in the deformed configuration $\mathbb{B} \subset \mathbb{R}^3$ at time $t$. The deformation gradient $\mathbf{F}$ with respect to the reference configuration $\mathbb{B}_0$ is given by

$$\mathbf{F}(\mathbf{X}, t) = \nabla \phi(\mathbf{X}, t) = \frac{\partial \phi(\mathbf{X}, t)}{\partial \mathbf{X}}, \quad \det \mathbf{F} > 0$$

In an appropriate kinematic framework for large deformation inelastic analysis including thermal effects and the macroscopic dilatation typical of damage, the deformation gradient is decomposed into thermal, plastic, and elastic parts as follows:

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\vartheta; \quad \det \mathbf{F}^p > 0, \quad \det \mathbf{F}^e > 0, \quad \det \mathbf{F}^\vartheta > 0$$

where $\mathbf{F}^e$ is the elastic deformation gradient, $\mathbf{F}^p$, the plastic deformation gradient and $\mathbf{F}^\vartheta$ is the thermal part of the deformation gradient. A graphical representation of equation (2) is given in Figure 1. The above decomposition of the deformation gradient allows us to account for the effects of temperature on plastic slip as well as the volumetric damage effects on elastic response. With the above kinematic framework, notions of an intermediate thermally expanded hot unstressed configuration and that of an intermediate hot plastically deformed relaxed (unstressed) configuration are introduced. Assuming isotropic thermal expansion, the evolution of the intermediate thermally expanded unstressed configuration is given as follows:

$$\dot{\mathbf{F}}^\vartheta \mathbf{F}^\vartheta^{-1} = \beta \dot{\mathbf{I}}$$

where $\beta$ is the thermal expansion coefficient and $\mathbf{I}$ is the second-order identity tensor.

Remark 1. The following alternative kinematic framework for constitutive modelling can also be considered, $\mathbf{F} = \mathbf{F}^e \mathbf{F}^\vartheta \mathbf{F}^p$, $\det \mathbf{F}^p > 0$, $\det \mathbf{F}^e > 0$, $\det \mathbf{F}^\vartheta > 0$. In this framework, the relaxed plastically deformed configuration is termed cold unstressed configuration. This stress free configuration can be visualized as being obtained by a thermo-elastic unstressing process from the original configuration as opposed to a pure elastic unstressing using the kinematic framework as in equation (2).

$\det \mathbf{F}^p$ can be considered as a measure of internal damage. Damage in the form of void growth comes into play by taking into account the balance of mass for a porous material, i.e.

$$\det \mathbf{F}^p = \frac{1 - f_0}{1 - f}$$

where $f_0$ and $f$ represent the void volume fractions in the initial and deformed configurations, respectively.
From the polar decomposition of the elastic deformation gradient, we can write the following:

$$ F^e = R^e U^e $$  \hspace{1cm} (5)

In the constitutive equations to be defined here, logarithmic stretches are used as strain measures. The elastic strain denoted by $E^e$ is defined in the intermediate hot unstressed configuration as

$$ E^e = \ln U^e $$  \hspace{1cm} (6)

The corresponding conjugate stress measure $\bar{T}$ used in the model is the pullback of the Kirchhoff stress with respect to $R^e$,

$$ \bar{T} = \det(U^e) R^{eT} T R^e $$  \hspace{1cm} (7)

where $T$ represents the Cauchy stress.

The body is assumed to be a continuum despite the presence of microvoids. The internal state variables are taken to be $(s, f)$, where $s$ represents the scalar resistance to plastic flow offered by the matrix material.

It is well known that the equivalent tensile stress $\sigma_m$ of the matrix material should be defined implicitly in terms of the Cauchy stress and $f$ [1, 8]. With the assumption of isotropy, a particular form of this dependency is the following:

$$ \Phi = \Phi(\sigma_m, f, p, \mathcal{I}) = 0 $$  \hspace{1cm} (8)

where the dependence of the potential $\Phi$ on the stress $\bar{T}$ is restricted to its first and second invariants. Here, the norm $\mathcal{I}$ of the stress deviator $\bar{T}' = \bar{T} + pI$, is given by

$$ \mathcal{I} = \sqrt{\bar{T}' \cdot \bar{T}'} $$  \hspace{1cm} (9)
where \( p \) is the mean normal pressure, \( p = -\text{tr}\tilde{T}/3 \). In the limit of a fully dense solid, the function \( \Phi \) is such that the classical \( J_2 \) von Mises isotropic plasticity theory is recovered, i.e. the following condition holds:

\[
\lim_{\epsilon \to 0} \Phi = \left( \frac{(3/2)\epsilon^2}{\sigma_m^2} - 1 \right)
\]

The evolution of plastic flow is defined with the following normality rule:

\[
\tilde{D}^p = \text{sym} \left( \tilde{L}^p \right) = \dot{\gamma}\tilde{c}_T\Phi = \dot{\gamma} \left[ (\tilde{\partial}_\nu \Phi)N - \frac{1}{3}(\tilde{\partial}_\rho \Phi)I \right]
\]

where \( \tilde{L}^p = \tilde{F}^p\tilde{F}^{-1} \) and \( N = \tilde{T}'/\mathcal{S} \) is the direction of the stress deviator. The spin in the intermediate unstressed plastically deformed configuration is immaterial for isotropic materials and is assumed to vanish resulting in the following equations:

\[
\tilde{D}^p = \tilde{L}^p, \quad \tilde{W}^p = 0
\]

Such a flow rule was first proposed and used successfully by Weber and Anand in [11, 14] for isothermal isotropic plasticity within the limits of small elastic stretches.

**Remark 2.** Consideration of the second law of thermodynamics and using the balance of energy equation, lead to a reduced dissipation inequality in which the conjugate force to \( \tilde{T} \) is the ‘plastic strain rate’ \( \dot{\tilde{L}}^p = \tilde{D}^p = \tilde{U}^{e-1}\text{sym}(C^e\tilde{L}^p)\tilde{U}^{e-1} \). Such a plastic strain rate and the maximum dissipation postulate [15] lead to a flow rule of the form \( \tilde{D}^p = \text{sym}(\tilde{L}^p) = \dot{\gamma}\tilde{c}_T\Phi \). This form is rather inconvenient for numerical implementations due to the elastic couplings in the definition of \( \tilde{D}^p \). In the case of small elastic stretches, i.e. when \( \tilde{U}^e \approx \tilde{I} \), such a flow rule coincides with the simplified flow rule given by equation (11).

**Remark 3.** It is interesting to note that in the case of small elastic stretches, \( \tilde{F}^e \approx \tilde{R}^e \), the stress measure \( \tilde{T} \) becomes identical to the Piola–Kirchhoff stress \( \tilde{S} = \text{det}(\tilde{F}^e)\tilde{F}^{-1} \tilde{T}\tilde{F}^{-T} \). The flow rule mentioned in Remark 2 is then identical to the flow rule, \( \tilde{D}^p = \dot{\gamma}\tilde{c}_S\Phi \), where, \( \tilde{D}^p = \text{sym}(C^e\tilde{L}^p) \), is the symmetric part of \( \tilde{L}^p \) relative to the metric \( C^e \), as proposed by Simo and Miehe [3].

The parameter \( \dot{\gamma} \) in equation (11) is determined using the work equivalence relation \( \tilde{T}' \cdot \tilde{D}^p = (1 - f)\sigma_m \dot{\varepsilon}_m^p \) which results in the following:

\[
\dot{\gamma} = \left[ \frac{(1 - f)\sigma_m}{\tilde{T}'\tilde{\partial}_\nu \Phi + p\tilde{\partial}_\rho \Phi} \right] \dot{\varepsilon}_m^p
\]

The evolution of the equivalent tensile plastic strain in the matrix \( \dot{\varepsilon}_m^p \) is specified via uniaxial experiments as

\[
\dot{\varepsilon}_m^p = f(\sigma_m, s, \theta)
\]

and the evolution of the isotropic scalar resistance \( s \) is also obtained from experiments and has the form,

\[
\dot{s} = g(\sigma_m, s, \theta)\dot{\varepsilon}_m^p - \dot{r}(s, \theta)
\]
where \( \dot{r}(s, \theta) \) is the static recovery function. The evolution equation for the void fraction is essentially a statement concerning conservation of mass and has the form (see equation (4)): \[
\dot{f} = (1 - f) \text{tr}(\mathbf{D}^p) \tag{16}
\]
The void fraction \( f \) is an important parameter in the potential \( \Phi \) and along with temperature is responsible for the loss in the stress carrying capacity of the material. In the framework of the Gurson–Tvergaard–Needleman model, \( \dot{f} \geq 0 \) is not a restriction. This feature of the damage model allows the modelling of reverse stiffening, i.e. of void healing at compressive hydrostatic stress. The material is assumed to have an initial finite porosity to trigger the developments of defects.

The hyperelastic constitutive model used, is as follows [11]:
\[
\mathbf{T} = \mathbf{L}^e \left[ \mathbf{E}^e \right]
\tag{17}
\]
where the isotropic elastic moduli \( \mathbf{L}^e \) are given by
\[
\mathbf{L}^e = 2\mu \mathbf{T} + (\kappa - \frac{2}{3} \mu) \mathbf{I} \otimes \mathbf{I}
\tag{18}
\]
where \( \mu \) is the shear modulus, \( \kappa \) is the bulk modulus and \( \mathbf{T} \) the fourth-order identity tensor. In general, we allow the elastic moduli to be functions of \( f \) and \( QDCL \). Calculation of \( f \) dependent elastic properties for microporous isotropic materials can be found in terms of temperature-dependent elastic properties of the matrix material, for example in [16].

Finally, the evolution equation for the temperature field is obtained from the first law of thermodynamics. In the absence of external heat sources, the balance of energy in the current configuration takes the following form:
\[
\rho c \dot{\theta} = \dot{\mathcal{W}}_{\text{mech}} - \nabla \cdot \mathbf{q}
\tag{19}
\]
where \( c \) is the specific heat capacity per unit mass (generally a function of \( f \) and \( \theta \)) and \( \rho \) the density in the current configuration. An isotropic constitutive equation for the heat flux \( \mathbf{q} \) is considered (Fourier’s law) as follows:
\[
\mathbf{q} = -K \nabla \theta
\tag{20}
\]
where the conductivity \( K(f, \theta) \geq 0 \). The mechanical dissipation \( \dot{\mathcal{W}}_{\text{mech}} \) is usually specified in terms of the plastic power by the following empirical law:
\[
\dot{\mathcal{W}}_{\text{mech}} = \omega \mathbf{T} \cdot \mathbf{D}^p
\tag{21}
\]
where \( \omega \in [0.85, 0.95] \) is a constant dissipation factor that represents the fraction of the plastic work that is dissipated as heat. In this work, the non-dissipative latent heating is considered negligible in comparison to the mechanical dissipation \( \dot{\mathcal{W}}_{\text{mech}} \).

3. TIME INTEGRATION SCHEME FOR THE THERMAL-DAMAGE FINITE DEFORMATION MODEL

In the constitutive incremental problem, the variables \( (T, s, f, F^e) \) are determined at the end of the time step (time \( t_{n+1} \)), given the body configuration \( \mathbf{B}_{n+1} \) and the temperature field at time \( t_{n+1} \). The body configuration \( \mathbf{B}_n \), the temperature field at time \( t_n \) and the variables \( (T, s, f, F^e) \)
at the beginning of the time step \((t_n)\) are known. Thus the solution is advanced within the incremental solution scheme by integrating the flow rule, the evolution equation for the state variable \(s\) and the evolution equation for the void volume fraction \(f\) over a finite time step \(\Delta t = t_{n+1} - t_n\). Figure 2 represents a schematic which depicts the evolution of the current, plastic unstressed, and thermal configurations within the framework of the multiplicative kinematics given by equation (2). The time integration algorithm presented here is an Euler-backward time stepping scheme based on an earlier algorithm presented in [11] that was developed for isothermal, defect-free materials.

The evolution equation for \(\mathbf{F}^p\) is \(\dot{\mathbf{F}}^p \mathbf{F}^{p-1} = \dot{\mathbf{D}}^p\). Integration of this equation yields

\[
\mathbf{F}^{p}_{n+1} = \exp\left[\Delta t \dot{\mathbf{D}}^p_{n+1} \left(\mathbf{T}'_{n+1}, s_{n+1}, \theta_{n+1}\right)\right] \mathbf{F}^p_n
\]

(22)

Integration of the evolution equation (3) for the thermal configuration yields

\[
\mathbf{F}^{\theta}_{n+1} = \exp\left(\Delta t \dot{\theta}_{n+1} \mathbf{I}\right) \mathbf{F}^{\theta}_n
\]

(23)

The other relevant relative deformation gradients (refer to Figure 2) can be evaluated as follows:

\[
\mathbf{F}_T = \mathbf{F}^{\theta}_{n+1} \mathbf{F}^{\theta-1}_{n} = \exp\left(\Delta t \dot{\theta}_{n+1} \mathbf{I}\right)
\]

(24)

\[
\mathbf{F}_C = \mathbf{F}^{p}_{n+1} \mathbf{F}_T \mathbf{F}^{p-1}_{n} = \exp\left[\Delta t \dot{\theta}_{n+1} \mathbf{I} + \Delta t \dot{\mathbf{D}}^p_{n+1} \left(\mathbf{T}'_{n+1}, s_{n+1}, \theta_{n+1}\right)\right]
\]

(25)

Consider the polar decomposition of both sides in the following equation expressing equality between deformation gradients (see Figure 2):

\[
\mathbf{F}_e^e = \mathbf{F}^{e}_{n+1} \mathbf{F}_C
\]

(26)
where $F^e_*$ is the known trial deformation gradient given by

$$F^e_* = F^e F^e_n$$  \hspace{1cm} (27)

and the relative deformation gradient $F^r$ is defined as

$$F^r = F_{n+1} F_n^{-1}$$  \hspace{1cm} (28)

By arguments on the uniqueness of the polar decomposition \cite{11}, we obtain the following:

$$R^e_* = R^{e}_n + Q^{e}$$  \hspace{1cm} (29)

$$U^e_* = U^{e}_{n+1} F_C$$  \hspace{1cm} (30)

Taking logarithms of the above tensorial equation and using equation (25), the following equation can be derived:

$$f + Y V E^{e}_n + 1 = f + Y V E^e_* - f + Y V D p_{n+1} \left[ Q F F \dot{Q} C R_{n+1} + 1 \right]$$  \hspace{1cm} (31)

where the trial elastic strain is given as $E^e_* = \ln U^e_*$.

From the hyperelastic constitutive equation (17) and the flow rule (11), the following equations for the deviatoric and pressure parts of $T^e_{n+1}$ are derived:

$$T^e_{n+1} = T^e_* - 2 \mu \Delta t \dot{\gamma}^e_{n+1} \frac{\partial}{\partial \phi} \Phi_{n+1}$$ \hspace{1cm} (32)

$$p_{n+1} = p_* - \Delta t \dot{\gamma}^e_{n+1} \frac{\partial}{\partial p} \Phi_{n+1} + 3 \kappa \beta \Delta \theta$$ \hspace{1cm} (33)

where $T^e_*$, $p_*$ are the deviatoric and pressure parts of $T_* = \mathcal{S}^e [E^e_*]$, respectively, and $\Delta \theta = \theta_{n+1} - \theta_n$. From equation (32) and the definition of $N$ the following statement can be made about the directions of the deviatoric stresses:

$$N_{n+1} = N_* = \frac{T^e_{n+1}}{\mathcal{S}_e}$$ \hspace{1cm} (34)

where $\mathcal{S}_e$ is the norm of the trial deviatoric stress. As a result, the tensorial equation (32) degenerates to a scalar equation involving the norms of the respective stress quantities with

$$T^e_{n+1} = \eta_{n+1} T^e_*$$ \hspace{1cm} (35)

where $\eta_{n+1}$ is the radial return factor given by

$$\eta_{n+1} = \frac{\mathcal{S}_{n+1}}{\mathcal{S}_e}$$ \hspace{1cm} (36)

Using the Euler-backward operator on the evolution equations for the state variable (15) and the void volume fraction (16) along with the scalar form of equations (32) and (33), the following set of non-linear algebraic equations is obtained:

$$\mathcal{S}_{n+1} = \mathcal{S}_* - 2 \mu \Delta t \dot{\gamma}_{n+1}^e \frac{\partial}{\partial \phi} \Phi_{n+1}$$ \hspace{1cm} (37)

$$p_{n+1} = p_* - \Delta t \dot{\gamma}_{n+1}^e \frac{\partial}{\partial p} \Phi_{n+1} + 3 \kappa \beta \Delta \theta$$ \hspace{1cm} (38)

$$s_{n+1} = s_* + \Delta t g(s_m(n+1), s_{n+1}, \theta_{n+1})$$ \hspace{1cm} (39)

$$f_{n+1} = f_n - (1 - f_{n+1}) \Delta t \dot{\gamma}_{n+1}^e \frac{\partial}{\partial p} \Phi_{n+1}$$ \hspace{1cm} (40)

In addition to the above equations, the solution \( \Phi(\sigma_m(n+1), f_{n+1}, p_{n+1}, s_{n+1}) \) must satisfy the potential equation (8), i.e.

\[
\Phi(\sigma_m(n+1), f_{n+1}, p_{n+1}, s_{n+1}) = 0
\]  

This system of non-linear equations (37)–(41) is solved simultaneously for \( \sigma_m(n+1), f_{n+1}, p_{n+1}, s_{n+1} \) using a Newton–Raphson procedure. Due to the singularity of the potential function \( f_+ \) in equation (10) and in Gurson-based models at \( \sigma_m = 0 \):

At near zero stress values, evolution of defects in the material is precluded. Thus, equation (40) is expressed as \( f_+ = f_+ \), equation (38) as \( p_+ = p_+ + 3k\beta\Delta \theta \) and so the system to be solved is reduced to two non-linear algebraic equations (37) and (39) with two scalar unknowns, \( s \) and \( S \) (with \( \sigma_m \) eliminated by expressing it as a function of \( f, p \) and \( s \) from equation (41)). Equations (37) and (39) can now be solved in an identical manner to the one used to solve the corresponding two non-linear algebraic equations for a von Mises damage-free isotropic, hyperelastic viscoplastic material [11]. The two-level iterative technique implemented in [17] has been used here to solve equations (37) and (39) for \( S \) and \( s \).

The Cauchy stress is obtained by a push forward operation

\[
\mathbf{T}_{e+1} = \exp \left( \frac{P_{n+1}}{K} \right) \mathbf{R}_e \tilde{T}_{n+1} \mathbf{R}_e^T
\]  

where \( \tilde{T}_{n+1} \) is given by

\[
\tilde{T}_{n+1} = \eta_{n+1} \mathbf{I} - \bar{p}_{n+1} \mathbf{I}
\]  

The concluding step is to update \( \mathbf{F}_{e+1}^e \). Using equation (35) and the deviatoric part of the constitutive relation \( \mathbf{F}_e = \mathcal{D}^e[\mathbf{E}_e^e] \), the following expression can be derived:

\[
\lambda^j \left( \frac{\det \mathbf{U}_{e+1}^e}{(\det \mathbf{U}_e^e)^{1/3}} \right)^{1/3} = \left[ \frac{\lambda^j}{(\det \mathbf{U}_e^e)^{1/3}} \right]_{n+1}
\]  

where \( \lambda^j \) are the eigenvalues of \( \mathbf{U}_{e+1}^e \). One can also show the validity of the following relations:

\[
[\det \mathbf{U}_{e+1}^e]^{-1} = \exp \left( \frac{P_{n+1}}{K} \right)
\]  

\[
[\det \mathbf{U}_e^e]^{-1} = \exp \left( \frac{P_*}{K} \right)
\]  

The substitution of equations (45) and (46) in (44) gives an expression for \( \lambda^j \) or more importantly \( \mathbf{U}_{e+1}^e \). \( \mathbf{F}_{e+1}^e \) is thus updated as \( \mathbf{F}_{e+1}^e = \mathbf{R}_e \mathbf{U}_{e+1}^e \). As opposed to the isothermal deformation of a defect-free material [11], \( \det \mathbf{U}_{e+1}^e \neq \det \mathbf{U}_e^e \), due to the macroscopic dilatation caused by thermal effects and internal damage.

Remark 4. A time integration procedure similar to the one proposed above can be applied to the kinematic constitutive framework suggested in Remark 1. The important qualitative features of such a framework are briefly described here. The functional dependency of the plastic stretching rate \( \mathbf{D}^p \) can be expressed as \( \mathbf{D}^p = \mathbf{D}^p_{n+1}(\tilde{T}_{n+1}, s_{n+1}, \theta_0) \) which is strongly in contrast to equation (22) where \( \mathbf{D}^p_{n+1} \) depends on \( \theta_{n+1} \). Similar dependencies can also be written for the evolution of...
of the state variable given by equation (15) and the equivalent plastic strain evolution governed by equation (14). This temperature dependency is the main feature which distinguishes the time-integration scheme for such kinematic framework from that governed by equation (2). In the present context, using ‘cold’ plastically deformed unstressed (relaxed) configuration is thus equivalent to using an explicit temperature dependence in the time integration procedure. Such a framework is not recommended for Lagrangian formulations as it eliminates any temperature dependence of the mechanical response.

4. DERIVATION OF THE CONSISTENT MATERIAL LINEARIZED MODULI

To solve the global equilibrium equations, the consistent tangent moduli have to be computed. The driving force for the time integration scheme is \( f + Y V_n \) and thus the linearized moduli are here defined as follows:

\[
\mathcal{C} = \frac{\partial T_{n+1}}{\partial E^{e}_n} \quad (47)
\]

where using equations (32), (33), (37) and (38), one can express \( T_{n+1} \) as follows:

\[
T_{n+1} = 2 \mu \dot{E}^{e}_n + (J_{n+1} - J) N + \kappa \text{tr} E^{c}_n I + (p - p_{n+1}) I \quad (48)
\]

with \( E^{e}_n \) defining the deviatoric part of the trial strain \( E^{c}_n \). Based on equations (47) and (48), the calculation of the elastoplastic moduli requires evaluation of the fourth-order tensor \( \frac{\partial f + Y V_n}{\partial E^{e}_n} \) as well as of the second-order tensors, \( \frac{\partial J}{\partial E^{e}_n} \), \( \frac{\partial p}{\partial E^{e}_n} \), \( \frac{\partial J}{\partial E^{e}_n} \), \( \frac{\partial p}{\partial E^{e}_n} \), and \( \frac{\partial J}{\partial E^{e}_n} \).

Using the definition of \( J \) and \( p \) as the magnitude and pressure parts of \( J_{n+1} = [E^{c}_n] \), it is straightforward to show that the tensors \( \frac{\partial J}{\partial E^{e}_n} \) and \( \frac{\partial p}{\partial E^{e}_n} \) are given as:

\[
\frac{\partial J}{\partial E^{e}_n} = 2 \mu N + \frac{1}{\mu} \frac{\partial J}{\partial E^{c}_n} \frac{\partial f}{\partial E^{c}_n} \quad (49)
\]

\[
\frac{\partial p}{\partial E^{e}_n} = -\kappa I - \text{tr} E^{c}_n \frac{\partial J}{\partial E^{e}_n} \frac{\partial f}{\partial E^{c}_n} \quad (50)
\]

where all variables are referred to time \( t_{n+1} \) and the elastic properties are calculated at \( (\theta_{n+1}, f_{n+1}) \) (the subscript \( n+1 \) is omitted here for notational convenience). The right-hand side of equations (49) and (50) contains the unknown tensor \( \frac{\partial E^{e}_n}{\partial f_{n+1}} \). This tensor can be calculated together with \( \frac{\partial E^{e}_n}{\partial f_{n+1}} \) by linearization of the system of algebraic equations (37)–(41). For this purpose, this set of equations is expressed as follows:

\[
\mathcal{F}_1 = J_{n+1} - J + 2 \mu \Delta t_{n+1} \Psi_{n+1} = 0 \quad (51)
\]

\[
\mathcal{F}_2 = p_{n+1} - p + \kappa \Delta t_{n+1} \Lambda_{n+1} - 3 \kappa \beta \Delta \theta = 0 \quad (52)
\]

\[
\mathcal{F}_3 = s_{n+1} - s - \Delta t_{n+1} = 0 \quad (53)
\]

\[
\mathcal{F}_4 = f_{n+1} - f + (1 - f_{n+1}) \Delta t_{n+1} \Lambda_{n+1} = 0 \quad (54)
\]

\[
\mathcal{F}_5 = \Phi_{n+1} = 0 \quad (55)
\]
Differentiation of this non-linear system of equations with respect to $\tilde{E}_s^*$, results in a linear system in terms of the derivatives of $(\sigma_m, f, p, J, s)$ at time $t_{n+1}$ with respect to $\tilde{E}_s^*$:

$$
\begin{bmatrix}
\partial_{\sigma_m} F_1 & \partial_f F_1 & \partial_p F_1 & \partial_J F_1 & \partial_s F_1 \\
\partial_{\sigma_m} F_2 & \partial_f F_2 & \partial_p F_2 & \partial_J F_2 & \partial_s F_2 \\
\partial_{\sigma_m} F_3 & \partial_f F_3 & \partial_p F_3 & \partial_J F_3 & \partial_s F_3 \\
\partial_{\sigma_m} F_4 & \partial_f F_4 & \partial_p F_4 & \partial_J F_4 & \partial_s F_4 \\
\partial_{\sigma_m} F_5 & \partial_f F_5 & \partial_p F_5 & \partial_J F_5 & \partial_s F_5
\end{bmatrix}
\begin{bmatrix}
\partial_{\tilde{E}_s^*} \sigma_m \\
\partial_{\tilde{E}_s^*} f \\
\partial_{\tilde{E}_s^*} p \\
\partial_{\tilde{E}_s^*} J \\
\partial_{\tilde{E}_s^*} s
\end{bmatrix}
= 
\begin{bmatrix}
\partial_{\tilde{E}_s^*} \sigma_m \\
\partial_{\tilde{E}_s^*} f \\
\partial_{\tilde{E}_s^*} p \\
\partial_{\tilde{E}_s^*} J \\
\partial_{\tilde{E}_s^*} s
\end{bmatrix}
\begin{bmatrix}
\partial_{\tilde{E}_s^*} \sigma_m \\
\partial_{\tilde{E}_s^*} f \\
\partial_{\tilde{E}_s^*} p \\
\partial_{\tilde{E}_s^*} J \\
\partial_{\tilde{E}_s^*} s
\end{bmatrix}
(56)
$$

The void fraction dependence of the elastic/thermal properties is also accounted for in the calculation of the (known) matrix in the above system. Let us denote this matrix with $\mathbf{M}$. The right-hand side vector in equation (56) is observed (see equations (49) and (50)) to depend on $\partial_{f, \nu} f_{n+1}$ which is one of the unknowns in this linear system. As a result of the elastic properties being dependent on the void fraction, the matrix $\mathbf{M}$ above can be suitably modified to generate a solvable linear system of equations for $\partial_{\tilde{E}_s^*} \sigma_m, \partial_{\tilde{E}_s^*} f, \partial_{\tilde{E}_s^*} p, \partial_{\tilde{E}_s^*} J, \partial_{\tilde{E}_s^*} s$ at time $t_{n+1}$. Let us denote by $\mathbf{P}$, the inverse matrix that results from the modified $\mathbf{M}$ in this linear system. We can then write the following:

$$
\begin{align*}
\partial_{\tilde{E}_s^*} f &= 2\mu P_{21} N_s - \kappa P_{22} I \\
\partial_{\tilde{E}_s^*} p &= 2\mu P_{31} N_s - \kappa P_{32} I \\
\partial_{\tilde{E}_s^*} J &= 2\mu P_{41} N_s - \kappa P_{42} I
\end{align*}
(57) - (59)
$$

Equation (57) can now be substituted in equations (49) and (50) to calculate the terms $\partial_{\tilde{E}_s^*} J_s$ and $\partial_{\tilde{E}_s^*} p_s$.

The last term that is required for the calculation of $\mathbf{C}$ is the fourth-order tensor $\partial_{\tilde{E}_s^*} N_s$. Using the definition of $N_s$ (equation (34)) and the relation $\tilde{E}_s' = J_s N_s / 2\mu$, one can show that

$$
\frac{\partial N_s}{\partial \tilde{E}_s^*} = \frac{2\mu}{J_s} \left( \mathcal{J} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right) - \frac{2\mu}{J_s} N_s \otimes N_s
(60)
$$

Using equations (47)–(50) and (57)–(60), the final expression for the consistent tangent moduli $\mathbf{C}$ can be written as follows:

$$
\mathbf{C} = x_1 \mathcal{J} + x_2 \mathbf{I} \otimes \mathbf{I} + x_3 N_s \otimes \mathbf{I} + x_4 N_s \otimes N_s + x_5 \mathbf{I} \otimes N_s
(61)
$$

where the various scalar coefficients above are defined as follows:

$$
\begin{align*}
x_1 &= 2\tilde{\mu} \\
x_2 &= \kappa P_{32} - \frac{2\tilde{\mu}}{3} \\
x_3 &= -\kappa P_{42} \\
x_4 &= -2\tilde{\mu} + 2\mu P_{41} \\
x_5 &= -2\mu P_{31}
\end{align*}
(62) - (66)
$$

with the algorithmic shear moduli defined as $\tilde{\mu} = \eta \mu$ and with the values of $\mu$ and $\kappa$ calculated at $(\theta_{n+1}, f_{n+1})$. The tangent moduli are in general non-symmetric ($x_3 \neq x_5$) in contrast to the symmetric moduli obtained in classical $J_2$ plasticity using an Euler-backward integration scheme [11]. The consistent tangent moduli given in [11, 14] is a special case of the present moduli for a
non-porous material (see also equation (10)). Similar calculations to the ones presented above can also be found in [18] where a forward Euler scheme was used to solve the system of equations (51)–(55). Finally, note that the mechanical phase problem will be solved with a frozen temperature field (see Section 5) and as such the temperature dependence of the mechanical material parameters does not affect the functional form of the linearized moduli in equation (61).

We will conclude this section with the calculation of the temperature derivative of the plastic work rate. This term is needed in the Newton–Raphson scheme that is used to solve the thermal field equations. The derivative of the energy dissipated as heat \( W_{\text{mech}} \) with respect to the temperature increment \( \Delta \Theta \) is evaluated as follows:

\[
\frac{\partial W_{\text{mech}}}{\partial \Delta \Theta} = \omega \left[ (1 - f) \frac{\partial \varepsilon_{m}^{p}}{\partial \Delta \Theta} + (1 - f) \frac{\partial \sigma_{m}}{\partial \Delta \Theta} \varepsilon_{m}^{p} - \frac{\partial f}{\partial \Delta \Theta} \sigma_{m} \varepsilon_{m}^{p} \right]
\]

(67)

The term \( \frac{\partial \varepsilon_{m}^{p}}{\partial \Delta \Theta} \) is obtained by the differentiation of the equation defining the evolution of the plastic strain rate (equation (14)) as follows:

\[
\frac{\partial \varepsilon_{m}^{p}}{\partial \Delta \Theta} = \frac{\partial \varepsilon_{m}^{p}}{\partial \sigma_{m}} \frac{\partial \sigma_{m}}{\partial \Delta \Theta} + \frac{\partial \varepsilon_{m}^{p}}{\partial \varepsilon_{m}^{p}} + \frac{\partial \varepsilon_{m}^{p}}{\partial \Delta \Theta} \]

(68)

The other derivatives that also need to be computed are given as

\[
\frac{\partial \sigma_{m}}{\partial \Delta \Theta} = 4 \sum_{i=1}^{4} \theta_{1i} a_{i}
\]

(69)

\[
\frac{\partial f}{\partial \Delta \Theta} = 4 \sum_{i=1}^{4} \theta_{2i} a_{i}
\]

(70)

\[
\frac{\partial s}{\partial \Delta \Theta} = 4 \sum_{i=1}^{4} \theta_{5i} a_{i}
\]

(71)

where \( \theta_{ij} \) are the components of the matrix \( M^{-1} \) (without any of the modifications of \( M \) discussed in the paragraph after equation (56)) and the constants \( (a_1, a_2, a_3, a_4) \) which are obtained by the temperature differentiation of the non-linear set of equations (51)–(55) are

\[
a_1 = -2 \frac{\partial \mu}{\partial \Theta} \Delta t \dot{f}_{n+1} \Psi_{n+1} - 2 \mu \Delta t \frac{\partial \dot{f}_{n+1}}{\partial \Theta} \Psi_{n+1}
\]

(72)

\[
a_2 = -\frac{\partial \kappa}{\partial \Theta} \Delta t \dot{f}_{n+1} \Lambda_{n+1} - \kappa \Delta t \frac{\partial \dot{f}_{n+1}}{\partial \Theta} \Lambda_{n+1} + 3 \kappa \beta + 3 \kappa \frac{\partial \beta}{\partial \Theta} \Delta \Theta + 3 \frac{\partial \kappa}{\partial \Theta} \beta \Delta \Theta
\]

(73)

\[
a_3 = \Delta t \frac{\partial \dot{g}_{n+1}}{\partial \Theta}
\]

(74)

\[
a_4 = (1 - f_{n+1}) \Delta t \frac{\partial \dot{f}_{n+1}}{\partial \Theta} \Lambda_{n+1}
\]

(75)

5. SOLUTION SCHEME FOR THE FIELD EQUATIONS

A two-level iterative scheme is employed at each increment to solve the field equations governing conservation of linear momentum (mechanical step) and the conservation of energy (thermal step). Each iterative step involves the solution of a non-linear deformation problem at fixed (predicted)
temperature (Phase 1) followed by a heat conduction problem at fixed (predicted) configuration (Phase 2). Thus, the solution scheme used splits the coupled thermoplasticity field equations by solving for the deformation field and the temperature field in distinct phases as opposed to solving for the deformation and temperature fields simultaneously during each iteration. An alternative solution scheme based on an operator split (isothermal split) of the non-linear coupled initial value problem which leads to a two-step solution procedure is proposed in [3]. This solution scheme, however, involves the development of a unified radial return algorithm that updates the mechanical plastic parameters during the thermal phase. A description of the main features of the present solution scheme in the mechanical and thermal phases follows.

5.1. Solution scheme for the mechanical phase

In the case of the updated Lagrangian analysis, the reference configuration is updated at each time step. Thus, the reference configuration at time \( t_{n+1} \) will be \( B_n \) and the current configuration \( B_{n+1} \).

In this work, we assume that the processes are quasi-static and that the body is under equilibrium at all times. Let \( T \) be the Cauchy stress and \( b \), the body force field defined on the current configuration \( B_{n+1} \). Then, at a time \( t_{n+1} \), the equilibrium equations can be written as

\[
\nabla_{n+1} \cdot T + b = 0 \quad \forall \mathbf{x} \in B_{n+1}
\]

(76)

The subscript \((n+1)\) indicates that the divergence of \( T \) is defined in the current configuration \( B_{n+1} \). The equilibrium equations can also be expressed in the reference configuration \( B_n \) as

\[
\nabla_n \cdot P_r + f_r = 0 \quad \forall \mathbf{x} \in B_n
\]

(77)

where the Piola–Kirchhoff I stress \( P_r \) and body force \( f_r \) are obtained as

\[
P_r = \det F_r \mathbf{T} F_r^{-T}
\]

\[
f_r = \det F_r b
\]

(78)

In order to solve the equilibrium equations (together with the constitutive problem presented earlier), certain boundary conditions have to be specified at all times on the boundaries of \( B \).

Let us suppose that the configuration \( B_n \) at time \( t = t_n \) is known and is under equilibrium. Under the action of the external forces, the body deforms and occupies the configuration \( B_{n+1} \) at time \( t = t_{n+1} \). Equation (77) describes the equilibrium of the body at time \( t_{n+1} \) expressed in the updated reference configuration \( B_n \). Let \( \Gamma \) be the surface (\( \Gamma \subset \partial B_n \)) which corresponds to regions of the body that may potentially come into contact with the die.

The incremental quasi-static boundary value problem at time \( t = t_{n+1} \) is to find the incremental (with respect to the configuration \( B_n \)) displacement field \( \mathbf{u}(\mathbf{x}, t_{n+1}) \equiv \mathbf{u}_{n+1} \) that will satisfy equation (77). The weak form of this equation is written as

\[
\hat{G}(\mathbf{u}_{n+1}, \tilde{\mathbf{u}}) \equiv G(\mathbf{u}_{n+1}, \tilde{\mathbf{u}}) + G_c(\mathbf{u}_{n+1}, \tilde{\mathbf{u}}) = 0
\]

(79)

for each test vector field \( \tilde{\mathbf{u}}(\mathbf{x}) \), which is compatible with the kinematic boundary conditions, where

\[
G(\mathbf{u}_{n+1}, \tilde{\mathbf{u}}) = \int_{B_n} P_r \cdot \frac{\partial \tilde{\mathbf{u}}}{\partial \mathbf{x}} \, dV - \left( \int_{\partial B_{n+1}} \mathbf{i} \cdot \tilde{\mathbf{u}} \, dA + \int_{B_{n+1}} b \cdot \tilde{\mathbf{u}} \, dV \right)
\]

(80)
and

\[ G_c(u_{n+1}, \hat{u}) = \int_{\Gamma} (t_N \cdot \hat{u} + t_T \cdot \hat{u}) \, dA \quad (81) \]

Equation (79) is a mixed form of the principle of virtual work. The internal work is expressed in the reference configuration, whereas the external work is expressed in the current configuration where the applied surface tractions, \( \hat{t} \), and body forces, \( \hat{b} \), are given. \( t_N \) and \( t_T \) represent the normal and tangential tractions at the die due to contact (and friction). The contact work is calculated in the reference configuration \( B_n \).

To solve the non-linear equation (79) for \( u(x_n, t_{n+1}) \), we employ a Newton–Raphson iterative scheme. Let \( u^{(k+1)}_{n+1} \) and \( u^{(k)}_{n+1} \) be the displacement fields at the end of the \((k+1)\)th step and the \( k \)th step, respectively, during the Newton–Raphson iterative process. Then, the linearized form of equation (79) will be

\[ \tilde{G} (u^{(k)}_{n+1}, \hat{u}) + \frac{\partial \tilde{G}}{\partial u^{(k)}_{n+1}} (u^{(k+1)}_{n+1} - u^{(k)}_{n+1}) = 0 \quad (82) \]

\( \tilde{G} \) consists of two terms, \( G \) and \( G_c \). In the following sections, we will provide the linearization of these terms. The term \( G \) contains the internal work and the external work done by the body forces and surface forces not including the contact work at a die workpiece interface:

\[ dG = \int_{B_n} \frac{\partial \hat{u}}{\partial x_n} \, dV + d(\text{bodyforces + applied traction}) \quad (83) \]

The linearizations for the follower forces and body forces have been dealt with in [19] and only the linearizations of the internal work term are reported here. We can write the following:

\[ dP_r = d(\det F_r T_{n+1} F_r^{-T}) \]

The complete linearization of this term leads to the following [11]:

\[ dP_r = \det F_r \left\{ \text{tr}(dF_r F_r^{-1}) T_{n+1} - T_{n+1}(dF_r F_r^{-1})^T \right. \]

\[ -\text{tr} \left( \frac{1}{3K} [dE^e] \right) T_{n+1} + \exp \left( \frac{P_{n+1}}{\kappa} \right) R^e [dE^e] (R^e)^T \]

\[ + (dR^e (R^e)^T) T_{n+1} - T_{n+1}(dR^e (R^e)^T)^T \} F_r^{-T} \quad (84) \]

\( \mathcal{C} \) denotes the consistent tangent linearized moduli. These moduli are computed for the coupled damage and thermoplasticity constitutive model/integration scheme given in the previous section. The linearization of the contact work is presented in [14, 20] and is not repeated here. An augmented Lagrangian implicit approach to contact is used and thus a full linearization of the principle of virtual work including the computation of the consistent tangent moduli and the contributions of contact to the tangent stiffness is carried out. Finite element issues (mesh locking, pressure modes, etc.) related to the near-incompressible nature of deformation are taken care of by using an assumed strain model with an \textit{a priori} stabilization scheme [14, 21]. Further details on the implementation are not the focus of this work and can be found in [14].

5.2. Solution scheme for the thermal phase

The temperature evolution equations obtained in Section 3 can be expressed in the following form:

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (K \nabla \theta) + W_{\text{mech}}$$ (85)

An implicit Euler backward time-stepping scheme is used. A Galerkin finite element scheme is used and the discretized spatial weak form of the time discrete equation can be expressed as

$$\mathbf{H}_i = \int_{\Omega} \left[ \rho c N_i (\theta_{n+1} - \theta_n) + \Delta t K \nabla \theta_{n+1} \cdot \nabla N_i - \Delta t \frac{\partial W_{\text{mech}}}{\partial \theta} (\theta_{n+1}) N_i \right] dV + \int_{\partial \Omega} N_i \Delta t q_n dA = 0$$ (86)

where $N_i$ are the basis functions and $i$ spans over the element nodes. Here $q_n$ refers to the outward normal heat flux which may be specified by a Neumann boundary condition or a convective boundary condition over a part of the boundary, i.e.

$$q_n = q_o \quad \text{Neumann b.c.}$$ (87)

$$q_n = h(\theta_{n+1} - \theta_\infty) \quad \text{Convective b.c.}$$ (88)

where $h$ is the heat transfer coefficient and $\theta_\infty$ is the ambient temperature. The components of the elemental Jacobian $\mathbf{J}$ are thus computed as

$$\mathbf{J}_{ij} = \frac{\partial \mathbf{H}_i}{\partial \theta_j} = \int_{\Omega} \left[ \rho c N_i N_j + \Delta t K \nabla N_i \cdot \nabla N_j - \Delta t \frac{\partial W_{\text{mech}}}{\partial \theta} N_i N_j \right] dV + \int_{\partial \Omega} h \Delta t N_i N_j dA$$

$$+ \int_{\Omega} \left[ \frac{\partial \rho c}{\partial \theta} N_i (\theta_{n+1} - \theta_n) N_j + \Delta t N_j \frac{\partial K}{\partial \theta} \nabla \theta_{n+1} \cdot \nabla N_i \right] dV$$ (89)

where $\partial \Omega_c$ is the part of the boundary with convective boundary condition. The temperature derivative of the mechanical dissipation term $\frac{\partial W_{\text{mech}}}{\partial \theta}$ was calculated in the previous section.

5.3. Summary of the coupled solution scheme

The implicit global solution strategy for a single time increment employed involves the following steps. $\phi$ denotes the body configuration.

1. **Initialization**
   
   $k = 1$
   
   $\theta_{n+1}^k = \theta_n$
   
   $\phi_{n+1}^k = \phi_n$

2. **Mechanical phase**
   
   Solve for the deformation field $\phi_{n+1}^{k+1}$ by solving the non-linear kinematic equations described in Section 5.1 with the temperature field frozen at $\theta_{n+1}^k$.

3. **Thermal phase**
   
   Solve for the temperature field $\theta_{n+1}^{k+1}$ by solving the non-linear thermal field equations described in Section 5.2 on a fixed body configuration given by $\phi_{n+1}^{k+1}$.
4. Update

\[
\text{IF } \frac{\|\mathbf{e}_{n+1}^{k+1} - \mathbf{e}_{n+1}^k\|}{\|\mathbf{e}_{n+1}^k\|} \leq \text{TOL} \\
\phi_{n+1} = \phi_{n+1}^{k+1} \\
\theta_{n+1} = \theta_{n+1}^{k+1} \\
\text{EXIT} \\
\text{ELSE} \\
k = k + 1 \\
\text{GOTO 2} \\
\text{ENDIF}
\]

The implementation was carried out in a completely object-oriented environment using \textit{diffpack} [22]. These computational features of the present simulator will not be discussed here and such details can be found in [14, 23].

6. NUMERICAL EXAMPLES

A comprehensive set of numerical examples including some in the simulation of metal-forming processes is considered here to validate the proposed computational framework and numerical algorithm. The CPU requirements of the present object-oriented implementation is shown in a representative example. The computations were performed on an IBM RS-6000 workstation at the Cornell Theory Center. In all reported simulations 4-noded quadrilateral elements were used for both the thermal and deformation parts of the analysis. Stabilization and incompressibility issues for such elements were addressed in [14]. The strain incompatibility that arises from using the same interpolation for both parts of the analysis was considered negligible due to the fact that the calculated thermal strains were moderate using the current updated Lagrangian formulation.

6.1. Single finite element tension test under adiabatic conditions without damage

A single 4-noded quadrilateral insulated finite element is subjected to a constant extension rate of 10.0 s\(^{-1}\) to produce a homogeneous state of stress. The specimen is made of 1100-Al and is assumed to be defect-free and at an initial temperature of 673 K. The constitutive model is presented in [10]. The flow function \(f\) (equation (14)) is given as follows:

\[
f(\sigma_m, s, \theta) = A \exp \left( -\frac{Q}{R\theta} \right) \left[ \sinh \left( \frac{Q}{s} \right) \right]^{1/m} \tag{90}
\]

and the hardening function \(h\) (equation (15)) is given by

\[
h(\sigma_m, s) = h_0 \left| 1 - \frac{s}{s^*} \right|^a \tag{91}
\]

with

\[
s^* = \tilde{s} \left[ \frac{f(\sigma_m, s, \theta)}{A} \exp \left( \frac{Q}{R\theta} \right) \right]^n \tag{92}
\]
Table I. Material parameters for Al 1100-O at 673 K

<table>
<thead>
<tr>
<th>Material parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$1.90593 \times 10^7 \text{s}^{-1}$</td>
</tr>
<tr>
<td>$Q/R$</td>
<td>$2.1086 \times 10^4 \text{K}^{-1}$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>7.0</td>
</tr>
<tr>
<td>$m$</td>
<td>0.23348</td>
</tr>
<tr>
<td>$s_0$</td>
<td>29.7 MPa</td>
</tr>
<tr>
<td>$h_0$</td>
<td>1115.6 MPa</td>
</tr>
<tr>
<td>$a$</td>
<td>1.3</td>
</tr>
<tr>
<td>$\tilde{s}$</td>
<td>18.92 MPa</td>
</tr>
<tr>
<td>$n$</td>
<td>0.07049</td>
</tr>
<tr>
<td>$\mu$</td>
<td>20.2 GPa</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>66.0 GPa</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$26.5 \times 10^{-6}$</td>
</tr>
<tr>
<td>$K$</td>
<td>238.0 N/(s K)</td>
</tr>
<tr>
<td>$\rho c$</td>
<td>2.820 MN/(m$^2$ K)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Table II. Convergence characteristics of the solution scheme in a typical time step (Example 1)

<table>
<thead>
<tr>
<th>Iteration number</th>
<th>Mechanical phase</th>
<th>Thermal phase</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Displacement norm</td>
<td>Work norm</td>
</tr>
<tr>
<td>1</td>
<td>1.0000e - 02</td>
<td>1.0000e + 00</td>
</tr>
<tr>
<td>2</td>
<td>6.8859e - 04</td>
<td>9.9016e - 01</td>
</tr>
<tr>
<td>3</td>
<td>3.8098e - 08</td>
<td>4.4737e - 02</td>
</tr>
<tr>
<td>4</td>
<td>8.4896e - 13</td>
<td>2.4986e - 06</td>
</tr>
<tr>
<td>5</td>
<td>2.8869e - 16</td>
<td>5.5660e - 11</td>
</tr>
</tbody>
</table>

Table III. The error norms for a global coupling loop (Example 1)

<table>
<thead>
<tr>
<th>Global coupling loop number</th>
<th>Error norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.754838e - 04</td>
</tr>
<tr>
<td>2</td>
<td>1.489692e - 07</td>
</tr>
<tr>
<td>3</td>
<td>1.264370e - 10</td>
</tr>
</tbody>
</table>

The specific values of the mechanical and thermal parameters are given in Table I. The calculations were performed with two different extension strain increments of 1 and 5 per cent. The flow stress and temperature responses are shown in Figures 3 and 4, respectively. Table II shows the convergence characteristics that were obtained in the mechanical and thermal phases of the first coupling loop in a typical time step. Quadratic convergence in both the mechanical and thermal phases is observed. The error norms for the coupling loop are shown in Table III. This loop is performed over the mechanical and thermal equations until convergence is achieved. These results agree very well with the adiabatic results reported in [2].

6.2. Single finite element tension test under isothermal conditions with damage effects

This test was performed under isothermal conditions (673 K) for the same material as in the previous example. An initial porosity of $f_0 = 0.05$ was assumed.

The viscoplastic potential as proposed by Gurson–Tvergaard–Needleman was used in our calculations. The flow potential is given as

$$
\Phi = \left[ \frac{3}{2} \frac{\dot{\gamma}^2}{\sigma_m^2} - 1 + 2q_1 f^* \cosh \left( \frac{3}{2} q_2 \frac{P}{\sigma_m} \right) - (q_1 f^*)^2 \right] = 0 \quad (93)
$$

with $q_1 = 1.5$ and $q_2 = 1$. The function $f^*(f)$ was proposed to handle the loss of stress-carrying capacity associated with void coalescence. This is valid for void volume fractions larger than a
certain critical value \( f_c \), and is taken as

\[
f^* = \begin{cases} 
  f & \text{if } f \leq f_c \\
  f_c + (f_u^* - f_c)(f - f_c)/(f - f_c) & \text{otherwise}
\end{cases}
\]

(94)

where \( f_u^* = 1/q_1 \). The material loses all stress-carrying capacity when the void volume fraction is \( f_c \). The values \( f_c = 0.15 \) and \( f_c = 0.25 \) as suggested in [1] are used.

The convergence characteristics using the exact tangent moduli are shown in Table IV for a typical time step. The prescribed tolerances for the \( L_2 \) norm for the displacement error and the energy (plastic work) error were set at \( 10^{-08} \). Convergence is achieved in 5 iterations. The obtained stress/strain relation is given in Figure 5.

The same test was also performed using the form of tangent moduli that corresponds to a dense (non-porous) material as given in [11, 14]. The computation of such dense material moduli does not involve inversions of the matrix \( M \) during every iteration and was found to be computationally cheaper per iteration. However, the convergence characteristics using this approximation were not acceptable. Convergence to the prescribed norms was not obtained even after 10 iterations. It was also noted that using the symmetric part of the consistent tangent moduli as proposed in [9] did not significantly affect the convergence characteristics. The solution converged in 5 iterations in this case as well.
1588 A. SRIKANTH AND N. ZABARAS

Figure 6. Initial mesh for the tapered forging test (Example 3)

Table V. Material parameters for Fe–2 per cent Si at 1273 K

<table>
<thead>
<tr>
<th>Material parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$6.346 \times 10^{11} \text{ s}^{-1}$</td>
</tr>
<tr>
<td>$Q/R$</td>
<td>$3.7569 \times 10^4 \text{ K}^{-1}$</td>
</tr>
<tr>
<td>$\tilde{\varepsilon}$</td>
<td>3.25</td>
</tr>
<tr>
<td>$m$</td>
<td>0.1956</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>66.1 MPa</td>
</tr>
<tr>
<td>$h_0$</td>
<td>3093.1 MPa</td>
</tr>
<tr>
<td>$a$</td>
<td>1.5</td>
</tr>
<tr>
<td>$\tilde{\sigma}$</td>
<td>125.1 MPa</td>
</tr>
<tr>
<td>$n$</td>
<td>0.06869</td>
</tr>
<tr>
<td>$\mu$</td>
<td>37.23 GPa</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>194.44 GPa</td>
</tr>
</tbody>
</table>

6.3. Isothermal forging of a tapered disk

The effect of the compression of an axisymmetric tapered disk between parallel dies is examined. An experimental investigation of this process in [24] has shown the correlation between the distribution of micro-cavities and wedge cracks with the tensile hoop stress and tensile hydrostatic stress which develops in the material at the perimeter of the disk. The influence of temperature on the microstructure was found to be negligible for reasonable ram speeds (strain rates) [24]. The process is therefore studied here under isothermal conditions. The material is assumed to have an initial porosity of $f_0 = 0.05$. The symmetry of the problem allowed modelling one-fourth of the geometry and the initial finite element mesh is shown in Figure 6. The dies are assumed to be rigid and to simulate sticking friction a Coulomb friction coefficient of 1.0 was assumed. A nominal strain rate of $0.02 \text{ s}^{-1}$ was applied during the forging process. The matrix material chosen for this test was Fe–2 per cent Si at an initial temperature of 1273 K [4]. The flow and hardening functions have the same functional form as in the first example and are taken from [10]. The specific values of the mechanical parameters are given in Table V.

The deformed mesh after 38 per cent height reduction is shown in Figure 7. The pressure contours are shown in Figure 8. The outer regions of the disk have tensile hydrostatic stress whereas the inside regions have compressive hydrostatic stress. This sign change is due to the geometry of the disk and is expected also to have a significance on the micro-structure of the material in these regions. The void fraction distribution is shown in Figure 9. It is observed that near the surface of the cylinder the void volume fraction shows a marked growth, while at the centre
Figure 7. Deformed mesh for the tapered forging test after 38 per cent height reduction (Example 3)

Figure 8. Contours of pressure (in MPa) in the final forged product (Example 3)

Figure 9. Distribution of the void fraction in the final forged product (Example 3)

Figure 10. Evolution of the void volume fraction at a point on the surface of the workpiece at the equator position (Example 3)

damage healing has occurred. Figure 10 shows a plot of the porosity as a function of time for a point on the surface of the workpiece in the equator position. The force necessary to forge the workpiece versus the stroke is shown in Figure 11. The force is compared to that required to forge a workpiece made of dense (non porous) material. The strain rate in the matrix $\dot{\varepsilon}_m$ at an early
stage of deformation is shown in Figure 12. The results of our simulation agree very well with the experimental observations in [24] and with a simulation of the process in [4] using ABAQUS [25]. The numerical values for the pressure are however much smaller than those reported in [4]. The convergence characteristics of the proposed computational scheme in a typical time increment are shown in Table VII. The computing statistics of this calculation are shown in Table VI.
Table VI. Computing statistics for the taper forging (Example 3)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total process time</td>
<td>19.0 s</td>
</tr>
<tr>
<td>Time step</td>
<td>0.25 s</td>
</tr>
<tr>
<td>Number of time increments</td>
<td>76</td>
</tr>
<tr>
<td>Displacement error $L_2$ norm</td>
<td>1.0e-06</td>
</tr>
<tr>
<td>Energy error $L_2$ norm</td>
<td>1.0e-06</td>
</tr>
<tr>
<td>Average Newton–Raphson iterations per increment</td>
<td>5.84</td>
</tr>
<tr>
<td>Formation of the linear system of equations</td>
<td>2.21 s</td>
</tr>
<tr>
<td>Time to solve the linear system</td>
<td>&lt; 0.5 s</td>
</tr>
<tr>
<td>Total CPU time</td>
<td>00:23:24</td>
</tr>
</tbody>
</table>

Table VII. Convergence characteristics in a typical time increment in the taper forging example

<table>
<thead>
<tr>
<th>Iteration number</th>
<th>Displacement error norm</th>
<th>Energy error norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.2922e-03</td>
<td>1.0000e+00</td>
</tr>
<tr>
<td>2</td>
<td>1.2680e-03</td>
<td>9.9097e-01</td>
</tr>
<tr>
<td>3</td>
<td>3.3275e-04</td>
<td>3.9061e-02</td>
</tr>
<tr>
<td>4</td>
<td>5.1053e-05</td>
<td>1.9203e-02</td>
</tr>
<tr>
<td>5</td>
<td>1.2694e-06</td>
<td>2.6024e-03</td>
</tr>
<tr>
<td>6</td>
<td>8.0443e-10</td>
<td>5.3337e-05</td>
</tr>
</tbody>
</table>

6.4. Isothermal axisymmetric drawing of a porous material

Central bursts or Chevron cracks are internal defects which are often encountered during cold drawing or extrusion. In addition to the central bursts, the so-called fir tree effect is observed where cracks develop on the surface of the extruded billet. Here, we simulate an isothermal, frictionless drawing operation with a die of average semicone angle 15°, and a diameter reduction of 8.5 per cent. The functional form for the strain rate is a power-law model with material rate sensitivity $m$, and the matrix material is assumed not to harden:

$$f(\sigma_m, s) = \dot{\varepsilon}_0 \left( \frac{\sigma_m}{s} \right)^{1/m}, \quad \dot{\varepsilon}_0 = 10^{-3} \text{ s}^{-1}, \quad m = 0.05, \quad s = 150 \text{ MPa}$$

(95)

The material is assumed to have an initial porosity of $f_0 = 0.01$. It was drawn with a nominal displacement rate of 0.01 s$^{-1}$. The symmetry of the problem allowed modelling one-half of the geometry. The initial and deformed finite element meshes are shown in Figure 13. Figure 14 shows the contours of the equivalent strain rate of the matrix and the material is observed to be deforming plastically primarily in the reduction region of the die. The pressure profile and the void fraction distribution are also shown and are extremely informative. It is noted that substantial negative pressure develops along the axis of the billet promoting void growth in these regions. The increase in porosity along the axis of the billet, after the material leaves the reduction region is as much as...
83 per cent. It is further noticed that the material is subject to a positive pressure when it comes in contact with the die in the reduction region and then to a negative pressure after it leaves the die. To our knowledge, this is the first time that a prediction of damage is reported in a process which is essentially contact driven, using a completely implicit contact algorithm (augmented Lagrangian formulation [20]). These results agree very well with the results in [4] obtained using ABAQUS.

6.5. Axisymmetric upsetting of a porous thermally insulated billet

This example is concerned with the simulation of axisymmetric upsetting and has been used as a benchmark problem in the isothermal case [11, 14]. The billet is cylindrical, with an initial radius of 1·0 mm and an initial height of 3·00 mm. Due to the symmetry of the problem only one-quarter of the specimen is modelled. A high Coulomb friction coefficient is assumed leading to perfect stick between the rigid tool and the billet. The upsetting is continued until a height reduction of 50 per cent is obtained with a nominal strain rate of 1·0 s$^{-1}$. The system is assumed
to be thermally insulated, preventing heat exchange from the workpiece to the environment. The process was studied for an initially porous \((f_0 = 0.05)\) 2024–T351 Aluminium alloy at an initial temperature of 300 K. The flow and hardening functions proposed by Lindholm and Johnson are obtained from [18] and have the following form. The flow function \(f\) is given as

\[
f(\sigma_m, s, \theta) = \dot{\varepsilon}_0 \exp \left[ \frac{1}{C} \left( \frac{\sigma_m}{s(\theta_m - \theta)/(\theta_m - \theta_o))^{\eta} - 1 \right) \right]
\]  

(96)

and the state variable \(s\) is given explicitly as

\[
s(t) = A + B \left( \varepsilon_m^p \right)^\eta
\]  

(97)

where the equivalent plastic strain \(\varepsilon_m^p\) is defined as

\[
\varepsilon_m^p = \int_0^t f(\sigma_m, s, \theta) \, dt
\]  

(98)

The specific values of the mechanical and thermal parameters are given in Table VIII and are taken from [18]. Complete dependence of the elastic mechanical and thermal properties on temperature as well as on the void fraction are accounted for in the simulation. The thermoelastic properties are taken as follows:

\[
\mu_m = \mu_{m,0} \left( 1 - \bar{\xi}_1 \frac{\theta - \theta_o}{\theta_m} \right)
\]  

(99)

\[
\kappa_m = \kappa_{m,0} \left( 1 - \bar{\xi}_2 \frac{\theta - \theta_o}{\theta_m} \right)
\]  

(100)

Table VIII. Material parameters for 2024–T351 Al at an initial temperature of 300 K

<table>
<thead>
<tr>
<th>Material Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>263·28 MPa</td>
</tr>
<tr>
<td>(B)</td>
<td>421·72 MPa</td>
</tr>
<tr>
<td>(C)</td>
<td>0·015</td>
</tr>
<tr>
<td>(\dot{\varepsilon}_0)</td>
<td>0·577 s(^{-1})</td>
</tr>
<tr>
<td>(\theta_o)</td>
<td>300 K</td>
</tr>
<tr>
<td>(\theta_m)</td>
<td>775 K</td>
</tr>
<tr>
<td>(\bar{\xi}_1)</td>
<td>0·5</td>
</tr>
<tr>
<td>(\bar{\xi}_2)</td>
<td>0·333</td>
</tr>
<tr>
<td>(n)</td>
<td>2</td>
</tr>
<tr>
<td>(\mu_{m,0})</td>
<td>26 GPa</td>
</tr>
<tr>
<td>(\kappa_{m,0})</td>
<td>68 GPa</td>
</tr>
<tr>
<td>(\beta_m)</td>
<td>22·6 \times 10(^{-6})/K</td>
</tr>
<tr>
<td>(\rho_m)</td>
<td>2770 kg/m(^3)</td>
</tr>
<tr>
<td>(c_m)</td>
<td>875 J/(kg K)</td>
</tr>
<tr>
<td>(\omega)</td>
<td>0·90</td>
</tr>
</tbody>
</table>

Copyright © 1999 John Wiley & Sons, Ltd.  
\[ K = \left(1 - \frac{3}{2}f\right)K_m \]  \hspace{1cm} (101)
\[ c = (1 - f)c_m \]  \hspace{1cm} (102)

Here, the subscript \( m \) refers to the properties of the matrix material and the subscript \( 0 \) refers to properties specified at the reference temperature \( \theta_0 \) \cite{18}. The method in \cite{16} was used to calculate the elastic properties \( \mu = \mu(\theta, f) \) and \( \kappa = \kappa(\theta, f) \) of the porous material in terms of the elastic properties of the matrix material given above. The large plastic deformation causes a self-heating of the specimen primarily induced by plastic dissipation. There is no heat generation at the die-workpiece interface due to the assumption of sticking friction. Figure 15 shows the initial and the deformed finite element meshes. In the axial compression of a cylinder, friction at the contact surface retards the radial outward flow of material at these surfaces and leads to barreling of the original cylindrical surface. Experimental results in \cite{26} have shown the effect of friction on the deformation to fracture for sintered powder materials. The significant role that friction plays in influencing the damage quality of the final product is also demonstrated in the current simulation. In the frictionless case the deformation was homogeneous and significant pore closure was observed (90 per cent reduction in the void volume fraction). In this frictionless case, the final forged product is expected to have a sound metallurgical structure. However, in the more realistic simulation which accounts for friction, this was not the case. Figure 16 shows

Figure 15. Initial and deformed meshes for the upsetting process of an insulated porous material (Example 5)

![Figure 15](image)

Figure 16. Void fraction evolution at Nodes A, B and C versus the die stroke (Example 5)

![Figure 16](image)
the progressive evolution of the void volume fraction for three material points; (a) Node A — at the centre (interior) of the workpiece, (b) Node B — at the equator (surface) of the workpiece and (c) Node C — a point on the axis in contact with the die. At the centre of the specimen (Node A), the material undergoes axial compression combined with lateral flow as a result of which nearly complete void closure is observed. The void fraction at the equator regions (Node B) decreases initially, but as the curvature of the bulge becomes more pronounced with increasing deformation, the void fraction starts showing a significantly increasing trend. Essentially, the non-uniform deformation under these conditions results in secondary tensile stress in the circumferential direction at the equator. Increasing the curvature increases this tensile stress which results in void growth in these regions. Insignificant pore closure was observed at Node C, since this point was chosen in the dead metal zone showing that a compressive hydrostatic pressure alone cannot lead to the closure of voids. The distribution of void fraction in the final forged product is shown in Figure 17. The pressure contours after 50 per cent height reduction are shown in Figure 18 and the outer radii of the workpiece is observed to be subject to negative pressures (tensile) which may lead to fracture (occurrence of surface cracks in upsetting). The evolution of the temperature at the centre of the billet (Node A) is reported in Figure 19, which, given the short process time, is rather insensitive to the idealized thermal boundary conditions. Figure 20 shows the temperature distribution at time $t = 0.75s$. A significant temperature rise (200 K) is observed as a result of plastic work being dissipated as heat. Negligible temperature rise is observed in the dead metal zone. Finally, the variation of the shear modulus $\mu$ in the final forged product as a result of temperature and void fraction evolution is shown in Figure 21. A maximum variation of about 8 per cent from the initial value was observed in the shear modulus.
6.6. Axisymmetric closed die forging

The influence of damage on macroscopic deformation characteristics as well as on mechanical parameters is shown in this extremely practical application of axisymmetric isothermal closed die forging of a 1100-Al cylindrical workpiece. The workpiece is assumed to be initially voided with \( f_0 = 0.05 \). An effective forging process requires that these existing voids be eliminated if the forged product is to achieve reasonable mechanical property levels. Unlike conventional fully
dense materials, the plastic deformation of a voided material will undergo volume change. As will be shown, this macroscopic deformation response of voided materials poses practical challenges in the design of flashless closed die forging processes. The process was analysed for 2 preforms. The preforms were cylinders chosen such that the initial volume equals that of the die cavity. Preform A
had an initial radius of 6.31 mm and an initial height of 10.0 mm. Preform B had an initial radius of 7.0 mm and an initial height of 8.12 mm. A realistic friction coefficient of 0.1 is assumed at the die-workpiece interface. A nominal strain rate of 0.01 s⁻¹ was applied during the forging process. Due to the symmetry of the problem, one-fourth of the geometry was modelled. The deformed mesh at the final stage is shown together with the initial mesh for preform A and preform B in Figures 22 and 23, respectively. This process was simulated under identical conditions for a fully dense material in [14] for Preform A, where at the end of the forging process the material filled the die cavity exactly without the formation of flash. It is evident from Figures 22 and 23 that

Copyright © 1999 John Wiley & Sons, Ltd.

there is a significant overall decrease in the volume of the workpiece as a result of void closure and the workpiece no longer fills up the die cavity exactly. An accurate estimate of this volume change, which is process dependent, is necessary for the design of flashless closed die forging processes. The force required to forge the product for both the design options is shown in Figure 24. The void fraction distribution in the final product for preform A and preform B are shown in Figures 25 and 26, respectively. It is evident from Figure 24 that the required work in obtaining the final product is significantly smaller for preform B than for preform A, whereas the quality of the forged product obtained using preform B is not very desirable. The void fraction distribution in the forged product indicate that for preform A, the final void fraction has decreased significantly in all the regions of the workpiece. However, with preform B this was not the case with some regions in the outer radii of the forged product showing an increase in void fraction of about 6 per cent. The pressure contours at the final stage of deformation for preform A and preform B are shown in Figures 27 and 28, respectively and are quite different. The results clearly indicate the important role that the preform geometry plays in determining the quality of the forged product. As a result of the change in the initial geometry, the workpiece was subjected
6.7. Shear band formation in plane strain tension

The influence of thermal softening and void growth have been studied in [1]. These studies indicate that under these conditions in plane strain, the deformations localize into a shear band. With thermal softening alone, the initiation of localization is only slightly delayed as when both progressive microrupture and thermal softening are accounted for. These statements are further reinforced by our simulations. The initial mesh chosen to model one-quarter of the specimen and the deformed mesh after the onset of shear band are shown in Figure 29. The top boundary is pulled at a constant speed of 25 m/s, which gives a nominal strain rate of \( \approx 500 \text{s}^{-1} \) in the central gauge section. An initial imperfection to trigger localization phenomenon is not necessary in the simulation due to the choice of the specimen geometry. The specimen is assumed to be thermally insulated. The matrix material chosen for this test was 2024 – T351 Aluminium alloy.
at an initial temperature of 300 K (Example 5) [4]. The thermal properties and elastic mechanical properties are assumed to be constant in this simulation. The extension force required to apply the end displacement is shown in Figure 30 for the fully dense and initially porous material ($f_0 = 0.05$). The effect of porosity in decreasing the strength of the material and in decreasing the plane strain ductility of the material is clearly evident. The contours of equivalent plastic tensile strain rate, void fraction and temperature for the initially porous material are shown in Figure 31. It was observed that for the fully dense material, the shear band initiates at an inclination of approximately 44$^\circ$, while for the initially porous material this angle was slightly different at 40$^\circ$ due to the pressure dependent response of the material. Figures 32–34 show the evolution of the equivalent plastic strain rate, the temperature and the porosity through the course of deformation at two points, one at the centre of the specimen (Node A) and one outside the central region on the axis of symmetry (Node B). These figures clearly indicate the degree of localization, which is
often identified with the ratio of the strain rate in the shear band (Node A) to that outside (Node B) becoming arbitrarily large. It is also pointed out that accuracy of localization predictions is often influenced by the choice of discretization. The results of the simulation agree well with the results in [18] where this problem is studied in a much more detailed manner.

7. DISCUSSION AND CONCLUSIONS

The aim of this work was to formulate a fully implicit time integration scheme for finite deformations of isotropic, thermal, hyperelastic–viscoplastic microporous materials. Previous works in the field of modelling ductile damage are with either hypoelastic formulations, elasto-plastic formulations (neglecting strain rate effects), infinitesimal plasticity, isothermal deformations, semi-implicit
time-stepping schemes or a combination of the above. The paper proposes a new constitutive framework that extends the multiplicative decomposition of the deformation gradient into elastic and plastic parts, to include the deformation induced by the thermal field as well as the effects of damage on the deformation. This is also for the first time that an accurate prediction of damage in metal-forming processes which are essentially contact-driven, is obtained using an implicit contact algorithm. A two level iterative scheme was used at each increment to solve the field equations governing the conservation of momentum (mechanical step) and the conservation of energy (thermal step) for the coupled thermo-mechanical problem. Exact linearizations are performed for each of these solution steps and the excellent convergence characteristics of the non-linear simulator are shown in a number of representative examples. Important algorithmic aspects of the implementation are also described. Finally, the proposed algorithm is used to study failure in bulk forming processes like rod drawing, upsetting and closed die forging. The cracks that occur in metal
working processes can be grouped into three broad categories: (a) cracks at a free surface such as at the bulge in upsetting a cylinder, (b) cracks that develop in a surface, such as in extrusion and (c) internal cracks, such as Chevron cracks in drawn bars. By a comprehensive set of examples, we have demonstrated the ability to predict accurately these failure mechanisms in practical metal-forming situations. An important factor in achieving a successful forming operation without fracture is also the level of hydrostatic pressure achieved in the process which is also computed in the various examples. A short presentation of the dramatic effect of thermal softening on the shear band localization in a plane strain tension test was finally presented.

ACKNOWLEDGEMENTS

The work presented here was funded by NSF grant DMII-9522613 to Cornell University. The computing for this project was supported by the Cornell Theory Center. The presented simulator was developed using the object oriented FEM environment of dixVTpack. The academic license for using the various libraries of dixVTpack is appreciated. Technical discussions with Yangang Bao on various implementation aspects of the work are appreciated. This paper was prepared while the senior author (NZ) was on sabbatical leave at École Polytechnique, Palaiseau, France and supported in part by the CNRS.

REFERENCES


