A FAMILY OF INTEGRATION ALGORITHMS FOR
CONSTITUTIVE EQUATIONS IN FINITE DEFORMATION
ELASTO-VISCOPLASTICITY

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SUMMARY

A two parameter family of incrementally objective integration schemes is proposed for the analysis of a broad range of unified rate-dependent viscoplastic constitutive models in large deformation problems. A similar scheme can be applied to rate-independent solids as well. These algorithms are a generalization of the mid-point integration rule. Full linearization of the principle of virtual work is performed in an updated Lagrangian framework together with a calculation of the consistent linearized moduli. Some details of the finite element implementation are given for plane strain and axisymmetric problems. The method is compared with other objective integration schemes and is tested with several examples where large strains and rotations occur.

INTRODUCTION

In recent years, the finite element method has been successfully applied to the analysis of many elasto-plastic problems involving finite strains and rotations. McMeeking and Rice\textsuperscript{1} were the first to propose an updated Lagrangian formulation where the reference state is updated to coincide instantaneously with the current configuration. Their approach predicts configurations which gradually shift from equilibrium and works well only for small incremental strains and rotations. Since then, many corrective measures and different formulations have been proposed. A detailed account of the literature is given by Cheng and Kikuchi.\textsuperscript{2}

Rate-independent plasticity is used to approximate the behaviour of solids at low temperatures. A large number of publications on the FEM implementation of rate-independent plasticity is available, including Needleman,\textsuperscript{3} Osias and Swedlow,\textsuperscript{4} Lee et al.,\textsuperscript{5} Nagtegaal and De Jong\textsuperscript{6} and others. Several rate-dependent viscoplastic constitutive models with internal variables have also been proposed to account for strain hardening, rate sensitivity and recovery in a wide range of temperatures ranging from room temperature to the melting point.\textsuperscript{7,8} These models all have a general overall structure which consists of a flow law (for the plastic rate of deformation) and evolution equations for the internal state variables. The above rates are prescribed functions of only the current values of the stress and state variables. These so-called true viscoplastic models differ from rate-independent and other traditional rate-dependent models\textsuperscript{9} in that there is no yield condition and loading/unloading criterion and that plastic deformation occurs at any level of stress. Rate-dependent elasto-viscoplastic FEM formulations include those of Argyris et al.,\textsuperscript{10} Eggert and Dawson\textsuperscript{11} and Taylor and Becker.\textsuperscript{12}

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Received 12 April 1990

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Revised 30 November 1990
The computational problem associated with large deformation plasticity can be divided into two parts. The first part involves the calculation of the material state (stress and state variables) at the end of a time step, given the material state in the beginning of the step and the deformation gradient of the configuration at the end of the step with respect to that in the beginning of the step. The second part which computes the deformation field involves the development of a Newton–Raphson scheme from a linearized form of the principle of virtual work and includes the calculation of the linearized material moduli.

The evolution of stress in elasto-viscoplastic models is usually expressed in a rate form through an objective hypoelastic form of elasticity. The stresses are found by integrating the stress rates. Several methods have been discussed for the integration of constitutive equations in small deformation plasticity. Most of the algorithms are based on an explicit integration of the constitutive relations and involve proportioning, projection or sub-incrementation to ensure plastic admissibility.

Integration of constitutive models with large deformations requires that the numerically calculated stresses satisfy the same objectivity requirements as the stresses themselves. Incrementally objective integration schemes were given by Hughes and Winget, Pinsky et al., Rubinstein and Atluri, Hughes, Nagtegaal, Nagtegaal and Veldpaus, Nagtegaal and Rebele, Reed and Atluri, Lush et al., Weber et al. and others. The most frequent integration approach used in the current literature is the employment of the radial return mapping where rotational-neutralized stress and deformation measures are used, as suggested by Nagtegaal and Rebele, Hughes and Winget, and others.

To increase the convergence rate of large deformation viscoplastic FEM algorithms, the Newton–Raphson method is usually employed with a full linearization of the principle of virtual work and a calculation of the consistent material moduli. It has been emphasized that, in order to preserve the quadratic convergence rate of the Newton–Raphson scheme, it is important to maintain consistency between the tangent operator and the integration scheme employed.

The plan of this paper is as follows. At first, a brief review of unified constitutive equations in large deformation elasto-viscoplasticity is given together with their rotational-neutralized form. Then, the generalized integration scheme is motivated and the required kinematic assumptions are presented. The objectivity of the proposed algorithm is demonstrated. The derivation of the Jacobian and consistent linearized moduli, a summary of the algorithm and details of the FEM implementation are then discussed. Finally, examples with large strains and rotations are presented and discussed using several rate-dependent constitutive equations.

**REVIEW OF CONSTITUTIVE EQUATIONS FOR RATE-DEPENDENT ELASTOPLASTICITY**

A constitutive model consists of evolution equations for the Cauchy stress \( \mathbf{T} \) and an internal scalar (or tensorial state variable), \( s \), called the isotropic deformation resistance. Assuming a hypoelastic form of elasticity, the Jaumann rate, \( \dot{\mathbf{T}} \), of the Cauchy stress is given as a linear isotropic function of the elastic rate of deformation, \( \dot{\mathbf{D}}^e \), which is here assumed to be the difference of the total, \( \dot{\mathbf{D}} \), and plastic, \( \dot{\mathbf{D}}^p \), deformation rates, respectively.

Using the continuum mechanics notation of Gurtin, one can write the following evolution equation:

\[
\dot{\mathbf{T}} = \mathcal{L}^e [\dot{\mathbf{D}} - \dot{\mathbf{D}}^p]
\]  

(1)
with the elastic isotropic moduli \( L^e \) defined as

\[
L^e = 2G\mathcal{T} + (K - \frac{2}{3}G)I \otimes I
\]

and where \( G \) and \( K \) are the shear and bulk moduli, respectively, and \( \mathcal{T} \) and \( I \) denote unit fourth and second order tensors, respectively. The symbol \( \mathcal{T} \mathcal{T} \) is reserved for the dot product between any fourth order tensor and any second order tensor, \( \cdot \) is used for the dot product of any two second order tensors and \( \otimes \) is used for the dyadic product of two same order tensors.

A flow rule describing the plastic rate of deformation is defined as follows:

\[
\dot{D}^p = \sqrt{\frac{3}{2}} \dot{\varepsilon}^p N^p(T', \bar{\sigma})
\]

The unit direction \( N^p \) of the plastic rate of deformation is given using the normality rule as

\[
N^p(T', \bar{\sigma}) = \sqrt{\frac{3}{2}} T' \bar{\sigma}
\]

where \( T' \) is the deviatoric part of the Cauchy stress and the equivalent stress is defined as

\[
\bar{\sigma} = \sqrt{\frac{1}{3} T' \cdot T'}
\]

In addition, the equivalent plastic strain rate is prescribed as a function of the current values of equivalent stress and state variable

\[
\dot{\varepsilon}^p = f(\bar{\sigma}, s)
\]

To complete the flow rule for rate-dependent plasticity, the evolution equation of a scalar state variable takes the form

\[
\dot{s} = g(\bar{\sigma}, s)
\]

\[
g(\bar{\sigma}, s) = h(s)\dot{\varepsilon}^p
\]

where \( h(s) \) denotes a positive hardening function.

The selection of both functions \( g(\bar{\sigma}, s) \) and \( f(\bar{\sigma}, s) \) is based on phenomenological theories usually related to micromechanics and a broad number of different models can be found in several articles.\(^7\)\(^-\)\(^9\)

**A FAMILY OF INCREMENTALLY OBJECTIVE INTEGRATION SCHEMES**

Let us assume that at time \( t_n \), the configuration \( B_n \) of the body and the pair of variables \( (T_n, s_n) \) are known. At time \( t_{n+1} = t_n + \Delta t \), the continuum is assumed to occupy the known configuration \( B_{n+1} \). Then the problem is to determine the state \( (T_{n+1}, s_{n+1}) \) at each material point in \( B_{n+1} \), based on the given constitutive relations. From now on, \( F_{n+1} \) will denote the relative deformation gradient of the configuration \( B_{n+1} \) with respect to the configuration \( B_n \) (i.e. \( F_n = I \)).

From a numerical point of view, an integration algorithm should be consistent with the constitutive equations and numerically stable. In addition it should be numerically objective, a physical requirement expressing the fact that the algorithm has to be invariant with respect to superimposed rigid body motions.

**Review of objectivity requirements**

Two motions \( x(X, t) \) and \( x^*(X, t) \) are called objectively equivalent if and only if

\[
x^*(X, t) = Z(t)(x(X, t) - o) + c(t)
\]
where $\mathbf{o}$ is a fixed point in space and the rigid body rotation $\mathbf{Z}(t)$ and translation $\mathbf{c}(t)$ are functions of time alone, of which $\mathbf{Z}(t)$ is subject to

$$\mathbf{Z}^T \mathbf{Z} = \mathbf{I} \quad \text{and} \quad \det \mathbf{Z} = 1$$

(10)

Now, it can be shown that

$$\mathbf{F}^* = \frac{\partial \mathbf{x}^*}{\partial \mathbf{X}} = \mathbf{Z} \mathbf{F}$$

(11a)

$$\mathbf{L}^* = \dot{\mathbf{F}}^* \mathbf{F}^*^{-1} = \dot{\mathbf{Z}} \mathbf{Z}^T + \mathbf{ZLZ}^T$$

(11b)

$$\mathbf{D}^* = \mathbf{ZDZ}^T$$

(11c)

$$\mathbf{W}^* = \dot{\mathbf{Z}} \mathbf{Z}^T + \mathbf{ZWZ}^T$$

(11d)

where $\mathbf{F}, \mathbf{D}, \mathbf{W}$ and $\mathbf{L}$ represent the deformation gradient, rate of deformation, spin and velocity gradients in the $\mathbf{x}(\mathbf{X}, t)$ motion and $\mathbf{F}^*, \mathbf{D}^*, \mathbf{W}^*$ and $\mathbf{L}^*$ represent the corresponding star quantities, respectively. In summary, a tensorial quantity is said to be objective or material frame-indifferent, if in any two objectively equivalent motions it obeys the appropriate tensor transformation law as given below for a vector $\mathbf{a}$, second order tensor $\mathbf{B}$ and fourth order tensor $\mathbf{e}$:

$$\mathbf{a}^* = \mathbf{Za}$$

$$\mathbf{B}^* = \mathbf{ZBZ}^T$$

$$\mathbf{e}^* = \mathbf{Z}[\mathbf{ZeZ}^T] \mathbf{Z}^T$$

Rotational-neutralized form of the constitutive model

Let us define the incremental rotation tensor $\mathbf{Q}(t)$, as the solution of the following initial value problem:

$$\dot{\mathbf{Q}}(t) \mathbf{Q}(t)^T = \mathbf{W}(t) \quad t_n \leq t \leq t_{n+1}$$

$$\mathbf{Q}(t_n) = \mathbf{I}$$

(12a)

(12b)

where the spin tensor $\mathbf{W}$ is defined as the antisymmetric part of the velocity gradient $\mathbf{L}$. The following bar transformations of $\mathbf{x}, \mathbf{F}, \mathbf{A}$ (any second order symmetric tensor), and $\mathbf{O}$ (any fourth order isotropic tensor) are also defined as

$$\bar{\mathbf{x}} = \mathbf{Q}^\top \mathbf{x}$$

(13a)

$$\bar{\mathbf{F}} = \mathbf{Q}^\top \mathbf{F}$$

(13b)

$$\bar{\mathbf{D}} = \mathbf{Q}^\top \mathbf{DQ}$$

(13c)

$$\bar{\mathbf{A}} = \mathbf{Q}^\top \mathbf{AQ}$$

(13d)

$$\bar{\mathbf{O}} = \mathbf{Q}^\top [\mathbf{Q}^\top \mathbf{OQ}] \mathbf{Q}$$

(13e)

These pull back (rotation-neutralized) quantities with $\mathbf{Q}$ would be used to define a convenient framework to perform the integration of the constitutive model defined by equations (1)–(8).

Indeed, following the previous definitions, the pull back Cauchy stress with $\mathbf{Q}$ can be defined as

$$\bar{T} = \mathbf{Q}^\top \mathbf{TQ}$$

(14)
and, as was shown by Nagtegaal and Veldpaus,\(^2\) equation (1) takes the following simplified form:

\[ \dot{T} = Q^T \dot{T} Q = \dot{\mathcal{D}}^e [\dot{\mathcal{D}}^e] = \dot{\mathcal{D}}^e [\mathcal{D} - \dot{\mathcal{D}}^p] \]  

(15)

with

\[ \dot{\mathcal{D}}^p = \sqrt{3} \dot{\varepsilon}^p \mathcal{N}^p (\dot{T}, \dot{\sigma}) \]  

(16a)

\[ \mathcal{N}^p (\dot{T}, \dot{\sigma}) = \sqrt{\frac{3}{2}} \frac{\dot{T}}{\dot{\sigma}} \]  

(16b)

\[ \dot{\sigma} = \sqrt{\frac{3}{2}} \dot{\mathcal{T}} \cdot \dot{T} \]  

(16c)

\[ \dot{\varepsilon}^p = f(\dot{\sigma}, s) \]  

(16d)

where for material obeying isotropic elasticity \( \dot{\mathcal{D}}^e = \mathcal{D}^e \).

Finally, the evolution equation for the state variable \( s \) remains the same as in equation (7).

**Generalized mid-point rule**

To derive the generalized mid-point integration rule, one can start from the following objective identity:

\[ \bar{T}_{n+1} = T_n + \int_{t_n}^{t_{n+1}} \dot{T} \, dt \]  

(17)

which, by using equation (15), can take the form

\[ \bar{T}_{n+1} = T_n + \int_{t_n}^{t_{n+1}} \mathcal{L}^e [\dot{\mathcal{D}}] \, dt - \int_{t_n}^{t_{n+1}} \mathcal{L}^e [\dot{\mathcal{D}}^p] \, dt \]  

(18)

Using the generalized mid-point rule to calculate the two integrals on the right side of the equation above, one can write

\[ \bar{T}_{n+1} = T_n + \mathcal{L}^e [\Delta \bar{E} - \Delta \dot{\bar{E}}^p] \]  

(19)

where the strain increments \( \Delta \bar{E} \) and \( \Delta \dot{\bar{E}}^p \) are defined as

\[ \Delta \bar{E} = \bar{D}_{n+1} \Delta t \]  

(20a)

\[ \Delta \dot{\bar{E}}^p = \dot{\bar{D}}_{n+1}^p \Delta t \]  

(20b)

with \( 0 \leq (\alpha, \beta) \leq 1 \), and \( \bar{D}_{n+1}^p \) and \( \dot{\bar{D}}_{n+1}^p \) are defined as the total and plastic rate of deformation at some intermediate configurations \( B_{n+1} \) and \( \dot{B}_{n+1} \) between \( B_n \) and \( B_{n+1} \), respectively. As will be discussed later, such a generalized two parameter mid-point rule includes a scheme that is second order accurate in the integration of both the total rate of deformation and the plastic rate of deformation \(( \alpha = 0.5 \text{ and } \beta = 0.5 \)), and a scheme which is second order accurate in the integration of the total rate of deformation, while still maintaining the efficiency of the radial return method \(( \alpha = 0.5 \text{ and } \beta = 1 \).

Finally, upon calculation of \( \bar{T}_{n+1} \) from equation (19), one can calculate the stress \( T_{n+1} \) as

\[ T_{n+1} = Q_{n+1} \bar{T}_{n+1} Q_{n+1}^T \]  

(21)

To effectively use the above algorithm in order to calculate the stress \( T_{n+1} \) corresponding to a given \( F_{n+1} \), one should assume an objective interpolation scheme for the calculation of the rate of deformation \( \bar{D}_{n+1}^p \) and the rotation tensor \( Q_{n+1} \), and also a scheme for the iterative calculation
of $\mathbf{D}_{n+\beta}$. The first two of these issues are undertaken in the next section, while a discussion of the last one follows.

Calculation of the plastic rate of deformation $\mathbf{D}_{n+\beta}$ requires the knowledge of the material state $(\mathbf{T}_{n+\beta}, s_{n+\beta})$. Assuming an iterative approach, one can calculate the right side of equation (20b) using the following approximations for $\hat{T}_{n+\beta}$ and $s_{n+\beta}$:

$$s_{n+\beta} = s_n + \beta \tilde{s}_{n+\beta} \Delta t$$  \hfill (22a)
$$\tilde{s}_{n+\beta} = g(\tilde{s}_{n+\beta}, s_{n+\beta})$$  \hfill (22b)
$$\hat{T}_{n+\beta} = (1 - \beta)T_n + \beta \hat{T}_{n+1}$$  \hfill (22c)

where $\hat{T}_{n+1}$ is calculated through an iterative process, as will be explained later. Note that the generalized mid-point rule for the calculation of the plastic strain increment $\Delta \mathbf{E}^p$ may be considered as a return mapping algorithm which corrects the elastic predictor $\hat{T}_{n+1}$ along a flow direction evaluated at the mid-point $(\hat{T}_{n+\beta}, s_{n+\beta})$, where

$$\hat{T}_{n+1} = T_n + \mathcal{L}^{\mathcal{E}}[\Delta \mathbf{E}_{n+\beta}]$$  \hfill (23)

The above algorithm has a form similar to that proposed by Ortiz and Popov for rate-independent constitutive models.\textsuperscript{19} It is first order accurate in consistency with the constitutive equations to be integrated. It has also been proved that for rate-independent solids it is unconditionally stable for $\beta \geq 0.5$ and conditionally stable for $\beta < 0.5$.\textsuperscript{19} The choice $\beta = 0.5$ leads to optimal accuracy for small time steps. However, the numerical applications will show that, when large time steps are used, larger values of $\beta$ (for example $\beta = 1$) may prove to be necessary in order to achieve stable and accurate solutions.

**Kinematic approximations**

In the sense of Reed and Atluri,\textsuperscript{26} algorithm (19)–(20) will be incrementally objective when for the motion $x^*(X, t)$ defined earlier (equations (9)–(11)), the algorithm predicts that

$$T_{n+1}^* = Z_{n+1} T_{n+1} Z_{n+1}^T$$  \hfill (24a)

and

$$s_{n+1}^* = s_{n+1}$$  \hfill (24b)

To derive a family of incrementally objective integration schemes based on the generalized mid-point rule proposed earlier, an approximation of $\mathbf{D}_{n+\beta}$ and $\mathbf{Q}_{n+1}$ is required. This interpolation scheme must be such that

$$\mathbf{D}_{n+\beta} = \mathbf{D}_{n+\beta}$$  \hfill (25a)

and

$$\mathbf{Q}_{n+1}^* = Z_{n+1} \mathbf{Q}_{n+1}$$  \hfill (25b)

Indeed, from equations (19), (20a), (20b), (22) and (25a) it can be concluded that

$$\hat{T}_{n+1}^* = \hat{T}_{n+1}$$

or using the definition of the bar transformed tensors

$$Q_{n+1}^T T_{n+1}^* Q_{n+1}^* = Q_{n+1}^T T_{n+1} Q_{n+1}$$

which, using equation (25b), leads to equation (24a). The truth of equation (24b) is quite obvious.
It is then our task to generate interpolation schemes which satisfy equations (25a) and (25b) for any rotation \( Z(t) \) defined such that \( Z(t_n) = Z_n \). Since the deformation gradients are given in the beginning \( (F_n = I) \) and end \( (F_{n+1}) \) of the time step, our goal is to derive an interpolation scheme of the following form:

\[
\bar{D}_{n+\alpha} = \hat{D}_{n+\alpha}(F_{n+1}, \alpha) \quad \text{with} \quad \hat{D}_{n+\alpha}(I, \alpha) = 0
\]  

(26a)

and

\[
Q_{n+1} = \hat{Q}_{n+1}(F_{n+1}) \quad \text{with} \quad \hat{Q}_{n+1}(U_{n+1}) = I
\]  

(26b)

Certain restrictions have to be imposed on the functions \( \hat{D}_{n+\alpha} \) and \( \hat{Q}_{n+1} \) in order to satisfy the objectivity constrains given by equations (25a) and (25b). Since these two equations are true for any proper rotation \( Z \), let us select \( Z_{n+1} = R_{n+1}^T \), where the rotation \( R_{n+1} \) is defined from the polar decomposition of \( F_{n+1} \), i.e. \( F_{n+1} = R_{n+1} \ U_{n+1} \). Then equations (26a, b) and (25a, b) give

\[
\hat{D}_{n+\alpha}(F_{n+1}, \alpha) = \hat{D}_{n+\alpha}(R_{n+1}^T F_{n+1}, \alpha) = \hat{D}_{n+\alpha}(U_{n+1}, \alpha)
\]  

(27a)

and

\[
\hat{Q}_{n+1}(F_{n+1}) = R_{n+1} \hat{Q}_{n+1}(R_{n+1}^T F_{n+1}) = R_{n+1}
\]  

(27b)

Equations (27a) and (27b) define the final required interpolation forms. Note that \( \hat{D}_{n+\alpha} \) is a function only of the right stretch tensor at the end of time step \( U_{n+1} \) and \( \alpha \), while \( \hat{Q}_{n+1} \) is equal to the known rotation tensor \( R_{n+1} \). An alternative proof of this conclusion can also be given following the methodology of Nagtegaal and Veldpaus.\(^{24}\)

A derivation of the present interpolation scheme for \( \hat{D}_{n+\alpha} \) is now given. The rotation-neutralized configuration at time \( t_{n+\alpha} \) (see Figure 1) is assumed as

\[
\bar{x}_{n+\alpha} = (1-\alpha)\bar{x}_n + \alpha\bar{x}_{n+1}
\]  

(28)

where the rotation-neutralized configuration at time \( t_{n+1} \) is approximated as

\[
\bar{x}_{n+1} = x_n + \Delta t \bar{x}_{n+\alpha}
\]  

(29)

With the configuration \( B_n \) selected as the reference configuration (i.e. \( \bar{F}_n = I \)), one can derive from

Figure 1. Schematic of a deforming body depicting the 'rotation-neutralized' states as described by the assumed kinematic approximations.
equation (28) that
\[ \vec{F}_{n+1} = (1 - \alpha)I + \alpha\vec{F}_{n+1} \]  
(30)

From equation (29), one can derive that
\[ \vec{L}_{n+1} = \frac{1}{\Delta t} (\vec{F}_{n+1} - I)\vec{F}_{n+1}^{-1} \]  
(31)

Following equation (27b), the rotation \( Q_{n+1} = R_{n+1} \), which modifies equations (30) and (31) to the following form:
\[ \vec{F}_{n+1} = (1 - \alpha)I + \alpha U_{n+1} \]  
(32a)
\[ \vec{L}_{n+1} = \vec{D}_{n+1} = \frac{1}{\Delta t} (U_{n+1} - I)\vec{F}_{n+1}^{-1} \]  
(32b)

From the above equations it becomes clear that the proposed integration scheme falls in the category of equations (27a) and (27b), and so it is an objective scheme for any value of \( \alpha \). It is quite interesting to observe that the present interpolation for \( \alpha = 0.5 \) takes the following form:
\[ \vec{D}_{n+1/2} = \frac{2}{\Delta t} (U_{n+1} - I)(U_{n+1} + I)^{-1} \]  
(33a)

This form has been proposed independently by Weber et al.,\textsuperscript{28} where the objective interpolation scheme proposed by Nagtegaal and Veldpaus\textsuperscript{24} was modified. More specifically, Nagtegaal and Veldpaus\textsuperscript{24} proposed that
\[ \vec{D}_{n+1} = \frac{1}{\Delta t} \ln(U_{n+1}) \]  
(33b)

which was modified by Weber et al.,\textsuperscript{28} using a Padé approximation of \( \ln(U_{n+1}) \). In the end, they independently arrived at the approximation given by equation (33a). Also note that, for \( \alpha = 0 \), \( \Delta E_{n+a} = \vec{D}_{n+a} \Delta t \) becomes the incremental Biot strain, while for \( \alpha = 1 \), \( \Delta \vec{E}_{n+a} \) takes a form similar to an Eulerian strain.

**Summary of the constitutive algorithm**

Let us assume that the time step size \( \Delta t \), the integration scheme parameters \( \alpha \) and \( \beta \), the deformation gradient \( F_{n+1} \), the material state at time \( t_n \) (\( T_n \), \( s_n \)) and an error tolerance \( \text{TOLER} \) are known. Then the following integration scheme is proposed for the calculation of the state \( (T_{n+1}, s_{n+1}) \).

---

**Step 1** Compute \( R_{n+1} \) and \( U_{n+1} \) using the polar decomposition of \( F_{n+1} \)
\( F_{n+1} = R_{n+1} U_{n+1} \) and then set \( \vec{F}_{n+1} = U_{n+1} \)

**Step 2** Calculate \( \vec{F}_{n+2} \) using equation (32a)

**Step 3** Compute the total bar strain increment based on rates at \( t_{n+a} \) (equations (32b) and (20a))

**Step 4** Calculate the trial stress and mean normal pressure
\[ T_{n+1} = T_n + \sigma' \Delta \vec{E}_{n+a} \quad \text{and} \quad \bar{p}_{n+1} = -\frac{1}{3} \text{tr}(T_{n+1}) \]

**Step 5** Calculate the deviatoric trial elastic stress and equivalent tensile stress
\[ T_{n+1} = T_{n+1} + \bar{p}_{n+1} I \quad \text{and} \quad \tilde{\sigma}_{n+1} = \sqrt{\frac{1}{2} T_{n+1}^T \cdot T_{n+1}} \]
Step 6 (Optional: only if Perzyna’s model is used\(^9\))
if \( \tilde{\sigma}_{n+1}^{\text{trial}} < s_n \) then

Elastic deformation

\[
s_{n+1} = s_n
\]

\[
T_{n+1} = T_{n+1}^{\text{trial}} = R_{n+1} T_{n+1}^{\text{trial}} R_{n+1}^T
\]

The constitutive algorithm is complete.
else

Deformation is elastic–plastic. Proceed to the next step.

Step 7 (when \( \beta \neq 1 \)) Start iterations, \( i = 1 \)

initialize \( T_{n+1}^i = T_{n+1}^{\text{trial}} \)

\[
s_{n+1}^{i+1} = s_n
\]

(i) get approximate stresses at \( t_{n+\beta} \)

\[
\bar{T}_{n+\beta} = (1 - \beta) T_n + \beta T_{n+1}^i
\]

(ii) estimate \( \tilde{\sigma}_{n+\beta} = \sqrt{\frac{3}{2} \bar{T}_{n+\beta} \cdot \bar{T}_{n+\beta}} \)

(iii) calculate \( \Delta s = \Delta t g(\tilde{\sigma}_{n+\beta}, s_{n+\beta}) \);

\[
s_{n+1}^{i+1} = s_n + \beta \Delta s
\]

(iv) estimate \( \tilde{\sigma}_{n+\beta}^{\text{p}} = f(\tilde{\sigma}_{n+\beta}, s_{n+\beta}) \)

(v) find the direction of \( D_{n+\beta}^{\text{p}}, \)

\[
N_{n+\beta}^{\text{p}}(\bar{T}_n, \tilde{\sigma}) = \sqrt{\frac{3}{2}} \frac{(\bar{T}_{n+\beta} / \tilde{\sigma}_{n+\beta})}{(\bar{T}_{n+\beta} / \tilde{\sigma}_{n+\beta})}
\]

(vi) calculate the total elastic strain increment based on rates at the configuration, \( B_{n+\beta} \),

\[
\Delta \bar{E}_{n+\beta}^{\text{p}} = \sqrt{\frac{3}{2}} \Delta t \tilde{\sigma}_{n+\beta}^{\text{p}} N_{n+\beta}^{\text{p}}
\]

(vii) update stress,

\[
T_{n+1}^{i+1} = T_{n+1}^{\text{trial}} - \mathcal{L}^e[\Delta \bar{E}_{n+\beta}^{\text{p}}]
\]

(viii) check for convergence

\[
\text{ERR} = \frac{\| T_{n+1}^{i+1} - T_n \|}{\| T_{n+1}^{i+1} \|} \times 100
\]

if \( \text{ERR} > \text{TOLER} \) then

set \( i = i + 1 \) and repeat steps (i) to (viii);
else converged;

\[
T_{n+1} = R_{n+1} T_{n+1}^{i+1} R_{n+1}^T
\]

\[
s_{n+1} = s_n + \beta \Delta s.
\]

Step 8 When \( \beta = 1 \), use the radial return method\(^{27}\)

(i) solve iteratively for \( s_{n+1} \) and \( \tilde{\sigma}_{n+1} \)

\[
s_{n+1} = s_n - \Delta t g(\tilde{\sigma}_{n+1}, s_{n+1}) = 0
\]

\[
\tilde{\sigma}_{n+1} = \tilde{\sigma}_{n+1}^{\text{trial}} + \Delta t 3Gf(\tilde{\sigma}_{n+1}, s_{n+1}) = 0
\]
(ii) calculate the radial return factor

\[ \eta_{n+1} = \frac{\tilde{\sigma}_{n+1}}{\sigma_{n+1}^{\text{trial}}} \]

(iii) update stress

\[ \tilde{T}_{n+1} = \eta_{n+1} \tilde{T}_{n+1}^{\text{trial}} + \tilde{P}_{n+1} \]

\[ T_{n+1} = R_{n+1} \tilde{T}_{n+1} R_{n+1}^T \]

Note that, when \( \beta = 1 \), the trial stress \( \tilde{T}_{n+1}^{\text{trial}} \) has the same direction as the stress \( T_{n+1} \) and the plastic increment \( \Delta \tilde{E}_{n+1}^p \). In this case equation (19) takes the scalar form given by the second equation of the step 8(i) above. The two equations in step 8(ii) are solved simultaneously using a Newton–Raphson scheme. With a reasonable initial guess for \( \tilde{\sigma}_{n+1} \) and \( s_{n+1} \) the Newton–Raphson loop is rapidly convergent, typically in three or four iterations. For relatively small strain increments the scheme is initialized with \( \tilde{\sigma}_{n+1} = \tilde{\sigma}_n \) and \( s_{n+1} = s_n \). For large increments this leads to slow convergence, as did a purely elastic initial guess. For Perzyna’s model\(^9\) the algorithm used to determine the initial guess is the radial return mapping for rate-independent plasticity.\(^{13}\) Such a technique is found to be both rapidly convergent and inexpensive. However, when power law type of models of the form \( \tilde{\sigma} = f(\tilde{\sigma}, s) = A(\tilde{\sigma}/s)^p \) are involved with the exponent \( (p) \) a big number and a non-hardening material \( (s = \text{constant}) \), proper attention has to be given so that the involved computation in step 8(ii) does not lead to erroneous results. In this work an elastic predictor–multi-step radial corrector type of algorithm is employed such that the ratio \( (\tilde{\sigma}/s) \) remains smaller than a prescribed value for each radial correction step.

**VARIATIONAL FORMULATION AND SOLUTION PROCEDURE**

Let us suppose that the configuration \( B_n \) of the body at time \( t = t_n \) is known and under equilibrium. Then the incremental quasi-static boundary value problem at time \( t = t_{n+1} \) is to find the incremental (with respect to configuration \( B_n \)) displacement field \( u(x_n, t_{n+1}) = u_{n+1} \) such that

\[ G(u_{n+1}, \tilde{u}(x_n)) = \int_{B_n} S_{n+1} \cdot \frac{\partial \tilde{u}}{\partial x_n} dV - \int_{\partial B_{n+1}} \tilde{t} \cdot \tilde{u} ds - \int_{B_{n+1}} \tilde{b} \cdot \tilde{u} dV = 0 \]  

(34)

for each test vector field \( \tilde{u}(x_n) \), which is zero on the portion of the boundary where kinematic boundary conditions are applied. The above equation is a mixed form of the principle of virtual work. The internal work is expressed in the reference configuration \( B_n \) using the Piola–Kirchhoff I stress, \( S_{n+1} = (\det F_{n+1}) T_{n+1} F_{n+1}^{-1} T_{n+1}^T \), while the external work is expressed in the current configuration where the applied surface tractions, \( \tilde{t} \), and body forces, \( \tilde{b} \), are given.

In order to solve the above set of non-linear equations for the incremental displacement field \( u(x_n, t_{n+1}) \), an iterative scheme must be used. In the present work a Newton–Raphson scheme is adopted, which requires linearization of equation (34) about the last obtained solution for \( U_{n+1} \). Denoting as \( d\phi \) the differential change of any quantity \( \phi \), one can derive (Appendix I) the following linearized form of the principle of virtual work:

\[ dG = \int_{B_n} \left\{ R_{n+1} \tilde{H}^{ep} [d\Delta E] R_{n+1}^T + (dR_{n+1} R_{n+1}^{-1}) T_{n+1} + T_{n+1} (R_{n+1} dR_{n+1}) \right\} dV \]

\[ - T_{n+1} F_{n+1}^{-1} dF_{n+1} + \text{tr}(dF_{n+1} F_{n+1}^{-1}) T_{n+1} \cdot \left\{ \frac{\partial \tilde{u}}{\partial x_n} F_{n+1}^{-1} (\det F_{n+1}) \right\} dV \]

\[ - d(\text{forcing terms}) = 0 \]

(35)
The differential change in the rotation-neutralized incremental strain $d\Delta \bar{E}$ is derived in a way consistent with the kinematic assumptions made in the previous section and is given as

$$d\Delta \bar{E} = \bar{F}^{-1}_{n+1} \text{sym} \left[ U^{-1}_{n+1} \text{sym} \{ F^T_{n+1} dF_{n+1} \} \right] \bar{F}^{-1}_{n+1}$$  \hfill (36a)

In equation (35) the moduli $\tilde{\mathcal{N}}^{sp}$ are derived by linearizing the integrated constitutive equation for the rotation-neutralized stress tensor and are known as the linearized moduli. They relate the differential change in the rotation-neutralized stress to the corresponding change in the rotation-neutralized strain as follows:

$$d\tilde{T}_{n+1} = \tilde{\mathcal{N}}^{sp} [d\Delta \bar{E}]$$  \hfill (36b)

The derivation of these moduli is discussed in Appendix II.

**FINITE ELEMENT IMPLEMENTATION**

Constant strain triangular elements have been used for plane and axially symmetric problems. At the element level the incremental displacement at any point within the element can be expressed in terms of the nodal degrees of freedom as

$$\{ d\bar{u}_{n+1} \} = [N] \{ d\bar{u}_{n+1} \}$$  \hfill (37)

where $[N]$ is the shape function matrix and $\{ d\bar{u}_{n+1} \}$ is the differential increment in nodal displacements. In deriving a system of algebraic equations for $\{ d\bar{u}_{n+1} \}$, all terms in equation (35) are expressed in terms of $dF_{n+1} = \partial d\bar{u}_{n+1} / \partial x_n$, which in turn, using equation (37), is expressed in terms of the unknown nodal displacement $\{ d\bar{u}_{n+1} \}$.

The calculated global Jacobian and the material moduli have been implemented in a finite element context. The set of the linear algebraic equations resulting from equation (35) takes the following general matrix form:

$$([K] + [K_G] + [K_L])^{(k-1)} \{ d\bar{u}_{n+1}^{(k)} \} = \{ dR_{n+1} \}^{(k-1)}$$  \hfill (38)

The stiffness $[K]$ is associated with the first term in equation (35). The matrix form of the required material moduli is given in Appendix II for a wide class of rate-dependent constitutive models. The stiffness $[K_G]$ is the contribution to the stiffness from all the other terms in equation (35). It includes geometric non-linearity effects and rotational effects arising from the use of an objective constitutive model. The last stiffness contribution, $[K_L]$, arises from the linearization of the follower forces.$^{32}$

Finally, the unbalanced load $\{ dR_{n+1} \}^{(k-1)}$ is defined as

$$\{ dR_{n+1} \}^{(k-1)} = G(\bar{u}_{n+1}^{(k-1)}, \bar{u}(x_n))$$  \hfill (39)

In summary, the steps in the Newton–Raphson scheme used for the calculation of $\{ d\bar{u}_{n+1} \}$ are given as follows:

a. Initialize $(n + 1)$ configuration variables with that of the $n$th configuration and set $k = 1$.
b. Compute the global Jacobian $[dG]^{(k-1)} = ([K] + [K_G] + [K_L])^{(k-1)}$ and unbalanced load vector $\{ dR \}^{(k-1)}$.
c. Solve $[dG]^{(k-1)} \{ d\bar{u}_{n+1} \} = \{ dR \}^{(k-1)}$ for $\{ d\bar{u}_{n+1} \}$.
d. Compute $T_{n+1}^{(k)}$ and $s_{n+1}^{(0)}$.
e. Check for convergence, if not converged then set $k = k + 1$ and repeat steps b to e.
f. Check for the end of the analysis. If $t_{n+1} = t_{\text{final}}$ then stop.
g. Update the reference state and go to step a.
NUMERICAL EXAMPLES

Example 1: Simple shear with superimposed rigid rotation

In order to verify the objectivity and accuracy of the present numerical algorithm, a simple shear problem with superimposed rigid body rotation is solved using Perzyna's rate-dependent model as shown in Table I. The model parameters and the time scale are selected such that rate effects are negligible. The results are then compared with the analytical solution of a rate-independent elastic–perfectly plastic material. As discussed by Taylor and Becker, the first order power law with a very large fluidity constant gives the most accurate characterization of the rate-independent material behaviour where strain rates are relatively small.

A simple shearing motion in a rectangular co-ordinate system with origin o and orthonormal base vectors \( \{e_i, i = 1,3\} \) is described by

\[
x = p + (\dot{y}t)p_2 e_1
\]

or in component form:

\[
x_1 = p_1 + (\dot{y}t)p_2; \quad x_2 = p_2; \quad x_3 = p_3
\]

<table>
<thead>
<tr>
<th>Table I. Illustrative constitutive models and material parameters</th>
</tr>
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<tbody>
<tr>
<td><strong>Perzyna's model</strong></td>
</tr>
</tbody>
</table>
| \( f(\bar{\sigma}, s) = \begin{cases} 
\gamma (\bar{\sigma}/s - 1.0)^n & \bar{\sigma} \geq s \\
0 & \bar{\sigma} < s 
\end{cases} \) |
| \( g(\bar{\sigma}, s) = h(\bar{\sigma}, s)f(\bar{\sigma}, s) \) |
| where                                                        |
| fluidity constant, \( \gamma = 150 \)                       |
| \( E = 25 \) 000 MPa;                                        |
| \( h(\bar{\sigma}, s) = 0 \)                                  |
| \( s_0 = 50 \) 00 MPa                                        |
| **Power law**                                                |
| \( f(\bar{\sigma}, s) = \ddot{\varepsilon}_0(\bar{\sigma}/s)^{1/m}; \) |
| \( g(\bar{\sigma}, s) = h(\bar{\sigma}, s)f(\bar{\sigma}, s) \) |
| where                                                        |
| \( \ddot{\varepsilon}_0 = 0.002 \) s\(^{-1}\);              |
| \( E = 25 \) 000 MPa                                         |
| \( h(\bar{\sigma}, s) = 0 \)                                  |
| \( s_0 = 50 \) 00 MPa                                        |
| **Hyperbolic sine model**                                    |
| \( f(\bar{\sigma}, s) = A \{\sinh(\zeta \bar{\sigma}/s)\}^{1/m} \) |
| \( g(\bar{\sigma}, s) = h(\bar{\sigma}, s)f(\bar{\sigma}, s) \) |
| \( h(\bar{\sigma}, s) = h_0(1 - s/s^*)^{1/n} \) \( \text{Sign}\{1 - s/s^*\} \) |
| \( s^* = \frac{\bar{\sigma}}{f(\bar{\sigma}, s)/A} \)        |
| Material parameters for aluminium 1100-O at 673°CK            |
| \( A = 4.75 \times 10^{-7} \) s\(^{-1}\);                   |
| \( \zeta = 70 \)                                             |
| \( m = 0.23348; \)                                           |
| \( s_0 = 47.11 \) MPa                                        |
| \( h_0 = 1115.6 \) MPa                                       |
| \( \ddot{\varepsilon} = 18.92 \) MPa;                       |
| \( n = 0.07049 \)                                            |
| \( G = 20.2 \) GPa                                           |
| \( K = 66.0 \) GPa                                           |
where \( \mathbf{p} \) is the position vector of a material particle at time \( t = 0 \). The analytical solution for this case has been obtained by Spencer and Ferrier\(^{33} \) and is also discussed by Weber et al.\(^{28} \) The homogeneous motion with superimposed rigid rotation is given by

\[
\mathbf{x} = Q(t) \left[ \mathbf{p} + (\dot{\gamma} t) p_2 e_1 \right]
\]

where \( Q(t) \) is the rotation tensor.

The calculation was performed with constant shearing rate \( \dot{\gamma} = 0.2 \text{ s}^{-1} \) and constant angular speed \( \dot{\theta} = 2\pi \text{ rad/s} \), with different values of fixed time step and different combinations of integration scheme parameters \( \alpha \) and \( \beta \). The calculated motion is shown in Figure 2, and the results for the normalized components \( \bar{T}_{12}/k \) and \( \bar{T}_{22}/k \) versus the amount of shearing strain \( \gamma \) are given in Figures 3–4. Here \( k ( = s_0/\sqrt{3}) \) denotes the deformation resistance in shear.

A comparison between the analytical and numerical results for \( \bar{T}_{12} \) and \( \bar{T}_{22} \) versus angle of rotation is given in Figures 3(a) and 3(b), respectively using different values of \( \alpha \) and \( \beta \) corresponding to the increments in shear \( \Delta\gamma = 0.01 \) and in rotation \( \Delta\theta = 18^\circ \). For smaller increments the difference between these schemes is indistinguishable. However, with small increments, \( \alpha = 0.5 \) and \( \beta = 0.5 \) give a more accurate result. As the increment size increases the algorithm with \( \beta = 1.0 \) gives better results than any other choice of \( \beta \).

From objectivity’s point of view the best combination found is \( \alpha = 0.5 \) and \( \beta = 1.0 \), which gives very good results with very large increments of rotation. Figure 4 shows a comparison of the exact solution with the numerical results of this scheme using \( \Delta\theta = 180^\circ \) and \( \Delta\gamma = 0.1 \). The agreement with the exact solution is very good.

\[\text{Example 2: Creep analysis of a thick-walled cylinder under a constant radial displacement rate at the inner wall}\]

Here the numerical procedure is used to carry out the creep analysis of a thick-walled cylinder with a specified constant radial displacement rate at the inner wall. The stability of the integration

---

*Figure 2. Calculated motion using \( \alpha = 0.5, \beta = 1.0, \Delta\theta = 90^\circ \) and \( \Delta\gamma = 0.05 \)*
Figure 3(a). Normalized shear stress \( (T_{12}/k) \) versus the angle of rotation with different values of \( \alpha \) and \( \beta \) for simple shear with superimposed rigid rotation of a rate-independent, elastic–perfectly plastic material.

Figure 3(b). Normalized direct stress \( (T_{12}/k) \) versus the angle of rotation with different values of \( \alpha \) and \( \beta \) for simple shear with superimposed rigid rotation of a rate-independent, elastic–perfectly plastic material.

scheme under different combinations of \( \alpha \) and \( \beta \) is examined. The problem is assumed to have axially symmetric geometry and a traction free outer surface, and an initial inner radius \( a_0 \) and outer radius \( b_0 \), with \( b_0/a_0 = 2 \). The geometry and the prescribed boundary conditions are shown in Figure 5.

All the calculations reported in this example are based on an elastic rate-dependent plastic constitutive model in a power law form (see Table I). Numerical results are compared with an analytical solution for the reaction pressure and circumferential stress versus the displacement of the inner wall, which is available for an incompressible, non-hardening and rate-dependent rigid–plastic material.\(^{29}\)
Figure 4. Normalized shear and normal stresses versus the angle of rotation for $\alpha = 0.5$ and $\beta = 1.0$ for the cases $(\Delta \theta = 90^\circ, \Delta \gamma = 0.05)$ and $(\Delta \theta = 180^\circ, \Delta \gamma = 0.10)$

Figure 5. Thick-walled cylinder under prescribed displacement rate at its inner wall

A constant displacement rate $\left(\dot{U}/a_0\dot{e}_0\right)$ of 1.0 is imposed on the inner wall and a comparison between the analytical and numerical results for the normalized pressure $(p/s_0)$ and normalized circumferential stress $(\sigma_{\phi\phi}/s_0)$ versus the normalized displacement $\bar{U} = (U/a_0\dot{e}_0)$ is shown in Figures 6–8, where $s_0$ is the initial value of $s$, $\dot{e}_0 = (s_0/E)$ is a reference strain and $\dot{e}_0$ a reference strain rate given in Table I.

In Figures 6(a) and 6(b), the numerical results are shown for different $(\alpha, \beta)$ values and a fixed time step of 0.5 s. For $\alpha = \beta = 0.1$, the numerical solution oscillates and at even larger time steps the numerical solution becomes unstable. However, with the decreasing step size the oscillations damp out and stable and accurate solutions are achieved. For $\alpha = \beta = 0.25$, the numerical solution is stable. However, for a step size greater than 0.5 oscillation begins. For $\alpha$ and $\beta$ values greater than 0.5, the solutions are stable. With an increase in step size the solution remains stable but becomes increasingly inaccurate for $\beta = 0.5$ as compared to $\beta = 1.0$. 


Figure 6(a). Normalized pressure \((p/k)\) versus the small normalized displacement \((\bar{U} = (U_{x}/a_{0}E))\) using \(\Delta t = 0.5\) s and different \((\alpha, \beta)\).

Figure 6(b). Normalized circumferential stress versus the large normalized displacement \((\bar{U} = (U_{x}/a_{0}E))\) using \(\Delta t = 0.5\) s and different \((\alpha, \beta)\).

For a small step size, \(\alpha = \beta = 0.5\) gives a highly accurate result for normalized displacements greater than 3.0, as shown in Figure 7. The error between the exact and numerical results at the beginning for very small values of displacement is due to the fact that an elastic-plastic material model with Poisson’s ratio close to 0.5 is used whereas the exact solution is calculated assuming an incompressible rigid-plastic material. This initial error also depends upon the stability characteristics of the integration scheme.

As a result of numerical tests, it is found that \(\alpha = 0.5\) and \(\beta = 1.0\) gives a highly stable and reasonably accurate integration scheme over a wide range of time step sizes. In Figure 8 we show our numerical results using this scheme for the large normalized displacements. The large
Figure 7. Normalized pressure and circumferential stress versus the large normalized displacement ($\bar{U} = (U_s/a_0E)$) using $\Delta t = 0.5$ s and $\alpha = \beta = 0.5$

Figure 8. Normalized pressure ($p/k$) and circumferential stress versus the large normalized displacement ($\bar{U} = (U_s/a_0E)$) using $\Delta t = 50.0$ s, $\alpha = 0.5$ and $\beta = 1.0$

increment size should be noted. The numerical results are in very good agreement with the analytical solution. To improve the accuracy a sub-incrementation technique was employed, as was discussed earlier.

**Example 3: Upset forging of a cylindrical billet**

In this example, the compression of a cylindrical billet between parallel and perfectly rough dies is investigated using Anand’s single scalar rate-dependent hyperbolic sine model. A few years back, 60 per cent upset of a billet with a specified constitutive equation was analysed by fourteen
different investigators in order to test the validity of various numerical methods for the analysis of metal forming problems.\textsuperscript{34} Since then, this problem is considered as a test example for computational procedures in large deformation elasto-viscoplastic analyses. Lush\textit{ et al.}\textsuperscript{27} and Weber\textit{ et al.}\textsuperscript{28} solved the same problem with a hyperbolic sine constitutive model and compared their finite element results with experimental results.

The geometry and the material data are the same as in Weber\textit{ et al.}\textsuperscript{28} and are summarized in Table I and Figure 9. Owing to the symmetry of the problem only a quarter of the billet is modelled. It is assumed that there is complete sticking between the die and workpiece. The loading is applied by prescribing the displacements of the rigid die such that the height of the billet is reduced by 60 per cent in 90 s, corresponding to a nominal strain rate of 0.01 s\textsuperscript{−1}.

![Figure 9. Geometry and boundary conditions for the upset forging of a cylindrical billet](image)

![Figure 10(a). Deformed and undeformed mesh after 60 per cent reduction in height](image)
Figure 10(b). Exploded view of the finite element mesh in the corner where fold over occurred after 60 per cent reduction in height.

Figure 11. Load versus deflection curve for the upset forging of a cylindrical billet.

The finite element discretization involves cross triangular elements, which do not show any sign of over-constraint behaviour under the incompressibility constraint of fully plastic deformation. Near the corner, where fold over is expected to occur, a refined mesh is used to accommodate this behaviour. An automatic time stepping algorithm is employed such that the maximum increment in equivalent plastic strain is not greater than 0.05. A 60 per cent reduction in height is achieved in 121 steps.

The deformed mesh is shown in Figures 10(a) and 10(b) along with the undeformed mesh. A considerable radial expansion of the billet and fold over at the corner are shown. The load–deflection curve obtained in the analysis is shown in Figure 11, which is in excellent agreement with the numerical solution of Weber et al.28

Figures 12–13 show the contours of equivalent plastic strain and deformation resistance after a 60 per cent reduction in height.
CONCLUSIONS

In this paper a formulation, finite element implementation and evaluation of the performance of a family of objective integration schemes for the general class of isotropic, rate-dependent constitutive models are given. The proposed integration family includes a scheme which is second order accurate in the calculation of the total and plastic strain increments ($\alpha = 0.5$, $\beta = 0.5$). It also includes a radial return algorithm which is second order accurate in the calculation of the total strain increment ($\alpha = 0.5$, $\beta = 1.0$). The accuracy and stability of the proposed algorithms are tested in three examples: a simple shear test under superimposed rigid motion, an expansion of a cylinder under prescribed displacement rate at its inner wall, and finally, the upset forging of a cylindrical billet. To evaluate the performance of the proposed family of integration schemes three different types of rate-dependent constitutive models are used for several combinations of the parameters $\alpha$ and $\beta$. The models include Perzyna's model, a commonly used power law and Anand's unified single internal state variable model. Out of the several combinations of $\alpha$ and $\beta$, it is found that for a wide range of strain increments the most efficient and accurate algorithm is the one with $\alpha = 0.5$ and $\beta = 1.0$.

ACKNOWLEDGEMENTS

The computing of this project has been supported with a grant from the University of Minnesota Supercomputer Institute. The authors would like to thank the reviewer of the paper for his valuable comments and suggestions.
Appendix I

Linearized form of the principle of virtual work

Denoting the estimate of \( u_{n+1} \) at the kth iterations as \( u^{(k)}_{n+1} \), one can write equation (34) in linearized form as

\[
G(u^{(k-1)}_{n+1}, \bar{u}(x_n)) + \frac{\partial G}{\partial u_{n+1}} |_{u = u^{(k-1)}_{n+1}} \cdot (u^{(k)}_{n+1} - u^{(k-1)}_{n+1}) = 0 \tag{40}
\]

The above equation can be used for the calculation of \( u^{(k)}_{n+1} \), given \( u^{(k-1)}_{n+1} \). The derivative of \( G \) with respect to \( u_{n+1} \) is calculated at the present guess \( u^{(k-1)}_{n+1} \), and is known as the global Jacobian. To calculate this, one can take the variation of equation (34) with respect to \( u_{n+1} \), as shown below:

\[
dG = \int_{\Omega_n} \frac{\partial \bar{u}}{\partial x_n} \partial V - \int_{\partial \Omega_{n+1}} \bar{u} \cdot \bar{t} \, ds + \int_{\Omega_{n+1}} \bar{b} \cdot \bar{u} \, dv \tag{41}
\]

Only the linearization of the first term is given here whereas the linearization of the last term is described by Hibbit.\(^{32}\)

Using the definition of the Piola–Kirchhoff I stress, one can show that

\[
dS_{n+1} = d(\det F_{n+1}) T_{n+1} F^{-1}_{n+1} + (\det F_{n+1}) dT_{n+1} F^{-1}_{n+1} + (\det F_{n+1}) T_{n+1} dF^{-1}_{n+1} \tag{42}
\]

The main part of the following analysis is to express all differentials in the equation above in terms of \( dF_{n+1} \), which will then be expressed in terms of \( du_{n+1} \), as equation (40) requires.

Using the identities

\[
d(\det F_{n+1}) = (\det F_{n+1}) \text{tr}(dF_{n+1} F^{-1}_{n+1}) \tag{43a}
\]

\[
dF^{-1}_{n+1} = -F^{-T}_{n+1} dF^{-T}_{n+1} F^{-1}_{n+1} \tag{43b}
\]

equation (42) can be further simplified as

\[
dS_{n+1} = (\det F_{n+1}) \{ \text{tr}(dF_{n+1} F^{-1}_{n+1}) T_{n+1} + dT_{n+1} - T_{n+1} F^{-T}_{n+1} dF^{-T}_{n+1} \} F^{-1}_{n+1} \tag{44}
\]

The calculation of \( dT_{n+1} \) must be consistent with the integration scheme used earlier for the calculation of \( T_{n+1} \). With variation of equation (21) and after some tensor algebra one can show that

\[
dT_{n+1} = [dR_{n+1} R^T_{n+1}] T_{n+1} + R_{n+1} d\bar{T}_{n+1} R^T_{n+1} + T_{n+1} [R_{n+1} dR^T_{n+1}] \tag{45}
\]

To complete the linearization process in the equation above, it remains to calculate

\[
dR_{n+1} R^T_{n+1} ( = -dR^T_{n+1} R_{n+1} ) \text{ and } d\bar{T}_{n+1}
\]

Taking the variation of the relative deformation gradient \( F_{n+1}^2 = (R_{n+1} U_{n+1}) \), one can finally derive

\[
dR_{n+1} R^T_{n+1} = dF_{n+1} F^{-1}_{n+1} - R_{n+1} (dU_{n+1} U^{-1}_{n+1}) R^T_{n+1} \tag{46}
\]

To compute \( dU_{n+1} \) in terms of \( dF_{n+1} \), one can use the identity

\[
U_{n+1} + U_{n+1} U_{n+1} = F^T_{n+1} F_{n+1} \]

To derive that

\[
dU_{n+1} + U_{n+1} U_{n+1} U_{n+1} = 2U_{n+1} \text{ sym} [F^T_{n+1} dF_{n+1}] \tag{47}
\]
Introducing Biot's strain \( E_n^{B+1} = U_{n+1} - I \), we have
\[
U_{n+1}^{-1} = (I + E_n^{B+1})^{-1} = I - E_n^{B+1} + O((E_n^{B+1})^2) \tag{48}
\]
Substitution of equation (48) in equation (47) gives
\[
dE_{n+1} = E_n^{B+1} dE_{n+1} + dE_{n+1} E_n^{B+1} + O((E_n^{B+1})^2) + dU_{n+1} = 2U_{n+1}^{-1} \text{sym} \{F_{n+1}^T dF_{n+1}\} \tag{49}
\]
Taking the transpose of equation (49) and adding to equation (49), the following linearized approximation of \( dU_{n+1} \) is obtained:
\[
dU_{n+1} = \text{sym} \{U_{n+1}^{-1} \text{sym} \{F_{n+1}^T dF_{n+1}\}\} + O((E_n^{B+1})^2) \tag{50}
\]
Substitution of equation (50) in equation (46) gives the following approximation:
\[
dR_{n+1} R_{n+1}^T = -dR_{n+1}^T \text{sym} \{U_{n+1}^{-1} \text{sym} \{F_{n+1}^T dF_{n+1}\}\} U_{n+1}^{-1} R_{n+1}^T
\tag{51}
\]
To complete the calculation of the right hand side of equation (45) in terms of \( dF_{n+1} \), one must calculate \( dT_{n+1} \).

From the constitutive part, we have
\[
dT_{n+1} = \mu^{sp} [d\Delta E] \tag{52}
\]
where \( \Delta E \) is defined from equation (20a) and the calculation of the consistent elastoplastic moduli \( \mu^{sp} \) will be undertaken in Appendix II. As before, it will be necessary to express \( d\Delta E \) in terms of \( dF_{n+1} \). From equation (32b) and using the definition of \( E_n^{B+1} \), the following approximation of \( \Delta E \) is obtained:
\[
\Delta E = E_n^{B+1} \{I - \alpha E_n^{B+1} + O(E_n^{B+1})^2\} \tag{53}
\]
The differential of equation (53) can be shown to take the form
\[
d\Delta E = (I - \alpha E_n^{B+1}) dU_{n+1} (I - \alpha E_n^{B+1}) + O((E_n^{B+1})^2)
\]
which, using equations (32a), (50) and the definition of \( E_n^{B+1} \), is modified as
\[
d\Delta E = F_{n+1}^{-1} \text{sym} \{U_{n+1}^{-1} \text{sym} \{F_{n+1}^T dF_{n+1}\}\} F_{n+1}^{-1} \tag{54}
\]
Finally, substitution of equations (45), (51) and (52) in equation (44) leads to the following form:
\[
dG = \int_{\Omega_n} \left\{ R_{n+1} (\mu^{sp} [d\Delta E]) R_{n+1}^T + (dR_{n+1} R_{n+1}^T) T_{n+1} + T_{n+1} (R_{n+1} dR_{n+1}^T)
\right. \\
\left. - T_{n+1} F_{n+1}^{-1} dF_{n+1} + \text{tr} (dF_{n+1} F_{n+1}^{-1} T_{n+1}) \right\} \left\{ \frac{\partial u}{\partial x_n} F_{n+1}^{-1} (\text{det} F_{n+1}) \right\} dV
\tag{55}
\]
\[- d \text{(forcing terms)} = 0 \]

From the above equation and equation (54), it becomes clear that \( dG \) can be expressed in the form of equation (40) which iteratively can be used to calculate \( du_{n+1} \).
APPENDIX II

Linearized moduli for rate-dependent plasticity

To derive the linearized material moduli (equation (52)), we take the variation of equation (19) with respect to $u_{n+1}$ as follows:

$$d\hat{T}_{n+1} = \mathcal{L}^{ep} [d\Delta \tilde{E} - d\Delta \tilde{E}^p]$$

(56)

where, from equations (16a) and (20b), it follows that

$$d\Delta \tilde{E}^p = \sqrt{\frac{2}{3}} \Delta t (\dot{\varepsilon}_{n+\beta}^p \cdot d\tilde{\mathbf{N}}_{n+\beta}^p + \dot{\varepsilon}_{n+\beta}^p \tilde{\mathbf{N}}_{n+\beta}^p)$$

(57)

The equivalent plastic strain increment $\dot{\varepsilon}_{n+\beta}^p$ was defined in equation (16d) from which using the chain-rule and equation (22b), one can derive

$$d\dot{\varepsilon}_{n+\beta}^p = \beta \frac{\partial f}{\partial \tilde{T}} \left. \frac{\partial \hat{\sigma}}{\partial \tilde{T}} \right|_{n+\beta} \cdot d\hat{T}_{n+1} + \frac{\partial f}{\partial \mathcal{S}} \left. \frac{\partial \hat{g}}{\partial \mathcal{S}} \right|_{n+\beta} ds_{n+\beta}$$

(58a)

where $\partial \hat{\sigma}/\partial \hat{T}$ is the flow vector for $J_2$ plasticity and is related to the unit vector $\tilde{\mathbf{N}}^p$ as follows:

$$\frac{\partial \hat{\sigma}}{\partial \hat{T}} = \sqrt{\frac{2}{3}} \tilde{\mathbf{N}}^p$$

(58b)

Similarly, one can derive that

$$d\tilde{\mathbf{N}}_{n+\beta}^p = \beta \frac{\partial \tilde{\mathbf{N}}^p}{\partial \hat{T}} \left. \frac{\partial \hat{\sigma}}{\partial \hat{T}} \right|_{n+\beta} [d\hat{T}_{n+1}]$$

(60)

Variation of equation (22a) leads to

$$ds_{n+\beta} = \beta \Delta t ds_{n+\beta}$$

which, using equations (7) and (22b), leads finally to the following expression for $ds_{n+\beta}$:

$$ds_{n+\beta} = \frac{\beta^2 \Delta t \left\{ \frac{\partial g}{\partial \tilde{T}} \left. \frac{\partial \hat{\sigma}}{\partial \tilde{T}} \right|_{n+\beta} \cdot d\hat{T}_{n+1} \right\}}{1 - \beta \Delta t \frac{\partial g}{\partial \mathcal{S}} \left. \frac{\partial \hat{g}}{\partial \mathcal{S}} \right|_{n+\beta}}$$

(59)

Finally, substituting equation (59) in equations (58a) and (58b), and then equations (58a), (58b) and (60) in equations (57) and (54), and taking common terms together yields

$$d\hat{T}_{n+1} = \left. \frac{\partial \hat{T}}{\partial \Delta \tilde{E}} \right|_{n+1} [d\Delta \tilde{E}] = \mathcal{M}^{ep} [d\Delta \tilde{E}]$$

(61)

where

$$\mathcal{M}^{ep} = \left[ \mathcal{L}^{ep^{-1}} + \sqrt{\frac{3}{2}} \beta \Delta t \left\{ \sqrt{\frac{3}{2}} \left( \frac{\partial f}{\partial \tilde{T}} \right)_{n+\beta} + \frac{\beta \Delta t \frac{\partial f}{\partial \mathcal{S}} \left. \frac{\partial \hat{\sigma}}{\partial \tilde{T}} \right|_{n+\beta}}{1 - \beta \Delta t \frac{\partial g}{\partial \mathcal{S}} \left. \frac{\partial \hat{g}}{\partial \mathcal{S}} \right|_{n+\beta}} \right\} \tilde{\mathbf{N}}_{n+\beta}^p \otimes \tilde{\mathbf{N}}_{n+\beta}^p \right]^{-1}$$

(62)
Note that, in order to calculate \( \tilde{\mathbf{N}}_{n+\beta}^p \) and \( (\partial \tilde{\mathbf{N}}^p / \partial \mathbf{T})_{n+\beta} \) in the equation above, one must assume the stresses \( \mathbf{T}_{n+1} \), and so an iterative scheme is required.

From equation (62)

\[
\mathbf{M}^{ep} = \left[ \mathbf{L}^{ep^{-1}} + \frac{3}{2} \beta \Delta t \left\{ \lambda \tilde{\mathbf{N}}_{n+\beta}^p \otimes \tilde{\mathbf{N}}_{n+\beta}^p + \tilde{\mathbf{e}}_{n+\beta}^p \frac{\partial \tilde{\mathbf{N}}^p}{\partial \mathbf{T}} \right\} \right]^{-1}
\]

where

\[
\lambda = c_1 + \frac{\beta \Delta t c_2 d_1}{1 - \beta \Delta t d_2};
\]

\[
c_1 = \left. \frac{\partial f}{\partial \sigma \tilde{\mathbf{e}}_{n+\beta}} \right|_{n+\beta}; \quad c_2 = \left. \frac{\partial f}{\partial \sigma \mathbf{s}_{n+\beta}} \right|_{n+\beta}; \quad d_1 = \left. \frac{\partial g}{\partial \sigma \tilde{\mathbf{e}}_{n+\beta}} \right|_{n+\beta}; \quad d_2 = \left. \frac{\partial g}{\partial \sigma \mathbf{s}_{n+\beta}} \right|_{n+\beta}.
\]

Now, using equation (16b) we have

\[
(\tilde{\mathbf{N}}^p)^T = \frac{1}{\sqrt{2\tilde{\sigma}}} \left\{ \tilde{T}_{11}' \tilde{T}_{22}' \tilde{T}_{33}' \tilde{T}_{12}' \tilde{T}_{21}' \tilde{T}_{31}' \right\}
\]

thus

\[
\tilde{\mathbf{N}}^p \otimes \tilde{\mathbf{N}}^p = \frac{3}{2\tilde{\sigma}^2} \mathbf{X}
\]

where for plane and axially symmetric problems

\[
\mathbf{X} = \begin{bmatrix}
(\tilde{T}_{11})^2 & (\tilde{T}_{11}' \tilde{T}_{22}) & (\tilde{T}_{11}' \tilde{T}_{12}) & (\tilde{T}_{11}' \tilde{T}_{33}) \\
(\tilde{T}_{22})^2 & (\tilde{T}_{22}' \tilde{T}_{12}) & (\tilde{T}_{22}' \tilde{T}_{33}) \\
\text{sym} & (\tilde{T}_{12})^2 & (\tilde{T}_{12}' \tilde{T}_{33}) & (\tilde{T}_{33})^2
\end{bmatrix}
\]

and

\[
\frac{\partial \tilde{\mathbf{N}}^p}{\partial \mathbf{T}} = \frac{\partial}{\partial \tilde{T}'} \left( \frac{3}{2} \tilde{T}' \right) = \sqrt{\frac{3}{2\tilde{\sigma}}} \left( \frac{\partial \tilde{T}'}{\partial \tilde{T}} - \tilde{\mathbf{N}}^p \otimes \tilde{\mathbf{N}}^p \right)
\]

with

\[
\mathbf{Y} = \frac{\partial \tilde{T}'}{\partial \tilde{T}} = \begin{bmatrix}
\frac{3}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\
\frac{3}{2} & 0 & -\frac{1}{2} & 0 \\
\text{sym} & 1 & 0 & \frac{3}{2}
\end{bmatrix}
\]

For plane problems only the upper \( 3 \times 3 \) portion is employed while for axially symmetric situations the complete matrices are utilized.
Finally,

\[ \tilde{\Omega}^{ep} = \left[ \tilde{\Omega}^{e-1} + \frac{3}{2} \beta \Delta t \left\{ \frac{3}{2\tilde{\sigma}_{n+1}^\alpha} \left( \lambda - \frac{\tilde{\dot{e}}_{n+\beta}^\alpha}{\tilde{\sigma}_{n+\beta}^\alpha} \right) X_{n+\beta} + \frac{\tilde{\dot{e}}_{n+\beta}^\alpha}{\tilde{\sigma}_{n+\beta}^\alpha} Y \right\} \right]^{-1} \]  

(63)

The linearized modulus for the particular case of \( \beta = 1 \) takes a simplified form which depends only on the trial stress \( \tilde{T}_{n+1} \). In summary, the material modulus for this case takes the following form:

\[ \tilde{\Omega}^{ep} = \tilde{\Omega}^{e} - 2G(\eta_{n+1} - \mu)\tilde{N}_{n+1}^p \otimes \tilde{N}_{n+1}^p \]  

(64)

where

\[ \tilde{\Omega}^{e} = 2\tilde{G}\tilde{\Omega} + (K - \frac{2}{3}\tilde{G})\mathbf{I} \otimes \mathbf{I} \]

\[ \tilde{G} = \eta_{n+1}\tilde{G} \]

with

\[ \eta_{n+1} = \frac{\tilde{\sigma}_{n+1}^\alpha}{\tilde{T}_{n+1}} \]

\[ \mu = \frac{b_2}{(a_1 b_2 + b_1 a_2)} \]

and

\[ a_1 = 1 + 3G\Delta t \left. \frac{\partial f}{\partial \tilde{\sigma}} \right|_{n+1} \;
\]

\[ a_2 = 3G\Delta t \left. \frac{\partial f}{\partial \tilde{\sigma}} \right|_{n+1} \]

\[ b_1 = \Delta t \left. \frac{\partial g}{\partial \tilde{\sigma}} \right|_{n+1} \]

\[ b_2 = 1 - \Delta t \left. \frac{\partial g}{\partial \tilde{\sigma}} \right|_{n+1} \]

REFERENCES